OPTIMAL CONTROL OF A POPULATION-VARYING HETEROSEXUAL HIV/AIDS MATHEMATICAL MODEL

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Abstract: We propose a heterosexual HIV transmission model of variable population size. The model is proved to be well-posed both mathematically and epidemiologically. A threshold for the existence of the disease is established, together with the existence and asymptotic stability of equilibria. An optimal control problem is formulated in which four controls (sex education, screening, HIV prevention methods, and treatment of infectives) are administered. It is shown that there exist an optimal control set and the optimal controls are characterised by applying Pontryagin's maximum principle. We numerically solve the optimal control problem using the forward-backward sweep method and ODE45 solver. The control strategy causes an important decrease in susceptibles, unaware infectives and AIDS individuals. The strategy also causes an outstanding transfer of infectives to the HIV treatment class.

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Key Words: heterosexual HIV transmission; asymptotic stability; Lyapunov function; optimal control; Pontryagin's Maximum Principle; forward-backward sweep method

1. Introduction

Globally, the number of people living with HIV continue to increase (see [25],[26]). However, many persons infected with HIV are unaware of their infection. In

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2017, an estimated 9.2 million of the 36.9 million people did not know that they had the virus (see [26]). Furthermore, for those who are screened, access to ART remains low, especially in limited resources settings. For example, about 21.7 million people accessed ART in 2017 out of an estimated 36.9 million people living with HIV in the same year (see [26]).

Several articles have applied optimal control to specific diseases such as influenza, HIV, tuberculosis, malaria. Zhou et al. [32] study an optimal strategy for HIV multitherapy. The study establishes four types of optimal treatment strategies consisting of different combinations of reverse transcriptase inhibitors and protease inhibitors. Mallela et al. [15] develop a model that incorporates various aspects of treatments for HIV–TB co-infected individuals. Their analyses show that co-infection treatments do not influence neither the basic reproduction number nor stability of the disease-free HIV equilibria. They also suggest that, for disease eradication, single disease treatments need to be incorporated with co-infection treatments. Juusola and Brandeau [10] develop a model to assist public health decision makers ascertain how to obtain optimal investment of limited resources in HIV treatment and prevention.

There are several studies that have investigated optimal control strategies in the transmission and treatment of HIV (see [2], [6],[8],[9],[10],[11],[14],[19]). However, most of these studies have assumed symmetry in the sex structure, that is, between male and females. Moreover, some studies have considered constant population size models which are applicable to diseases where births and deaths are balanced or the time period of the disease is very short or disease-related deaths are insignificant (see [6],[9],[11],[19]). However, these assumptions fail to hold in the transmission dynamics of HIV/AIDS since the disease is endemic in humans whose communities are always of variable population sizes and the influence of disease mortality is clearly evident. For example, an estimated 940 000 people died from AIDS-related illnesses globally in 2017 out of about 36.9 million people who were living with HIV (see [25]).

Our study aims to minimise HIV infection through use of targeted sex education, prevention methods, screening of unaware infectives and treatment of HIV infectives. We analyse a two-sex HIV/AIDS transmission model with a variable population size in the presence of deaths related to the disease. Optimal control is applied to obtain the most optimal intervention strategy to reduce the number of HIV infectives while balancing the control efforts. We define screening to include testing and counselling for HIV and other Sexually Transmitted Infections (STI's) and screening for tuberculosis (TB). Antiretroviral therapy (ART) is used to both treat HIV positive individuals and to reduce the transmission risk to an uninfected sexual partner. World health Organisa-

tion (WHO) reports that risk of transmission to an uninfected partner by an HIV positive person who adheres to an effective ART combination can be reduced by 96% (see [25]).

HIV infection is greatly attributed to heterosexual transmission especially in Southern Africa which is the worst affected by the virus (see [26],[27]). Sex structured models may contribute meaningfully on how best to make use of limited resources in the HIV prevention, treatment and control. A gender asymmetry heterosexual model characterised by multiple concurrent partnerships is considered where rates are differentiated according to sex, e.g., susceptibility, screening, treatment and progression to AIDS (see [1],[3],[12]). In heterosexual transmission, sexual contact occurs only between partners of the opposite sex and we assume that only sexually active individuals are susceptible, that is, only horizontal HIV incidence is considered. The transmission dynamics of HIV is determined by the sexual behaviour of the infected individuals. Screening of unaware HIV infectives for treatment is important since we understand that ART reduces transmission of HIV.

In Section 2, we describe structure of heterosexual HIV/AIDS model under study. We discuss the basic properties of the model in Section 3. In Section 4, we formulate the optimal control problem, prove the existence and uniqueness of the optimal control, characterise the optimal control, and numerically compute the optimal control. Finally, conclusions are presented in Section 5.

2. Structure of the model

The total human population is divided into two interacting classes, namely, females f and males m . The female and male populations are further divided into homogeneous sub-populations, namely susceptibles $S_j(t)$, unaware infectives $I_{1j}(t)$, screened and aware infectives but not on treatment $I_{2j}(t)$, HIV positive individuals who are on treatment $H_j(t)$, and AIDS individuals $A_j(t)$ for $j = f, m$ where f and m stand for females and males respectively. We denote the total variable population sizes for females and males by $N_f(t)$ and $N_m(t)$ respectively. Here, $I_{2j}(t)$, $H_j(t)$ and $A_j(t)$ are new classes of infectives (with different infectivity) that arise once the disease has invaded the population. The HIV/AIDS model is as follows:

$$\frac{dS_f}{dt} = Q_f N_f - \beta_{mf} S_f - \mu_f S_f, \quad (1a)$$

$$\frac{dS_m}{dt} = Q_m N_m - \beta_{fm} S_m - \mu_m S_m, \quad (1b)$$

$$\frac{dI_{1f}}{dt} = \beta_{mf}S_f - (\theta_f + \delta_f + \mu_f)I_{1f}, \quad (1c)$$

$$\frac{dI_{1m}}{dt} = \beta_{fm}S_m - (\theta_m + \delta_m + \mu_m)I_{1m}, \quad (1d)$$

$$\frac{dI_{2f}}{dt} = \theta_f I_{1f} - (\psi_f + \mu_f + \pi_f)I_{2f}, \quad (1e)$$

$$\frac{dI_{2m}}{dt} = \theta_m I_{1m} - (\psi_m + \mu_m + \pi_m)I_{2m}, \quad (1f)$$

$$\frac{dH_f}{dt} = \pi_f I_{2f} - (\epsilon_f + \mu_f)H_f, \quad (1g)$$

$$\frac{dH_m}{dt} = \pi_m I_{2m} - (\epsilon_m + \mu_m)H_m, \quad (1h)$$

$$\frac{dA_f}{dt} = \delta_f I_{1f} + \psi_f I_{2f} + \epsilon_f H_f - (\mu_f + \alpha_f)A_f, \quad (1i)$$

$$\frac{dA_m}{dt} = \delta_m I_{1m} + \psi_m I_{2m} + \epsilon_m H_m - (\mu_m + \alpha_m)A_m, \quad (1j)$$

with positive initial conditions while all parameters are non-negative and total population $N(t) = N_f(t) + N_m(t)$, $N_j(t) = S_j(t) + I_{1j}(t) + I_{2j}(t) + H_j(t) + A_j(t)$, $j = f, m$ and $\frac{dN_j}{dt} = (Q_j - \mu_j)N_j - \alpha_j A_j$. The average number of adequate sexual contacts with female infectives per unit time of one susceptible male, β_{fm} is given by

$$\frac{\beta_{I_{1f}} c_{I_{1f}} b_{I_{1f}} I_{1f} + \beta_{I_{2f}} c_{I_{2f}} b_{I_{2f}} I_{2f} + \beta_{H_f} c_{H_f} b_{H_f} H_f + \beta_{A_f} c_{A_f} b_{A_f} A_f}{N_f}.$$

The average number of adequate sexual contacts with male infectives per unit time of one susceptible female is given by β_{mf} can be written in a similar way. β_{mf} and β_{fm} are variable functions of the total population for each sex since the population size is varying. The probability that an HIV individual will infect a susceptible depends primarily on the stage of infection and sexual behaviour of both the infective and the susceptible. Unaware infectives are considered to be in the acute HIV infection stage while aware (not on ART) infectives are in the chronic stage of the virus. Therefore we assume that $\beta_{I_{ij}} > \beta_{I_{2j}}$, $j = f, m$. Furthermore, we assume that $\beta_{A_j} > \beta_{I_{ij}} > \beta_{I_{2j}} > \beta_{H_j}$ for each $j = f, m$. The assumptions here are that individuals at the AIDS stage (if they recklessly behave) are the most infectiousness and HIV treatment reduces transmission to about 96% with good adherence (see [1],[16],[25]). $c_j > 0$ since infectives from any class are, on average, assumed to have sexual contact with other individuals of opposite sex. The rate at which unaware infectives are detected by screening efforts depends on the size of the population of unaware infectives

and on the prevalence of infection on each sex, that is, $\theta_j = \theta_j(I_{1j}, N_j)$. The AIDS individuals are assumed to be capable of transmitting the virus. Utilising our model, we can define the total number of new HIV infections, I_{new} , and disease-related deaths, A_D , as follows:

$$I_{new} = \int_0^T (\beta_{mf} S_f + \beta_{fm} S_m) dt, \quad (2a)$$

$$A_D = \int_0^T (\alpha_f A_f + \alpha_m A_m) dt, \quad (2b)$$

where T is the final time. The total HIV and AIDS burden can be represented by $k_1 I_{new} + k_2 A_D$ where k_1 and k_2 are weighted coefficients delegated to new HIV infections and disease-related deaths, respectively. New HIV infections and disease-related deaths have great importance in the effort to reduce the effects of HIV as a world epidemic (see [27]). However, there are several other expressions for the total burden of HIV and AIDS depending on the priorities of the studies (see [15]). The meaning of our parameters is shown in Table 1.

3. Basic properties of the model

3.1. Well-posedness of the model

Our interest is on states that are non-negative. It is elementary to show that non-negative initial conditions to the model (1a)-(1j) lead to non-negative solutions for all time $t \geq 0$ and will remain non-negative. Therefore the mathematically and epidemiologically feasible region for system (1a)-(1j) is given by

$$D = \left\{ \mathbf{X} \in \mathbf{R}_+^{10} \mid \begin{array}{l} S_f + I_{1f} + I_{2f} + H_f + A_f = N_f \\ S_m + I_{1m} + I_{2m} + H_m + A_m = N_m \end{array} \right\},$$

where $\mathbf{X} = (S_f, S_m, I_{1f}, I_{1m}, I_{2f}, I_{2m}, H_f, H_m, A_f, A_m)$.

3.2. Constant population size

The total populations $N_j(t)$, $j = f, m$ may remain constant when $\frac{dN_j}{dt} = 0$, that is, when $(Q_j - \mu_j)N_j - \alpha_j A_j = 0$. In this case, the number of AIDS individuals cannot be identically zero and the following restrictions are imposed: $Q_f = \mu_f$, $Q_m = \mu_m$ and $\alpha_j = 0 \forall j = f, m$. Our interest in this paper is to study the model when population is not constant and we assume that $Q_j \neq \mu_j$

Table 1: Description of parameters (all non-negative); $j = f, m$

Parameter	Symbol
Probability of transmitting HIV by unaware infective to a susceptible during sexual encounter	β_{I_1j}
Probability of transmitting HIV by aware infective not on ART during sexual encounter	β_{I_2j}
Probability of transmitting HIV by infective on ART during sexual encounter	β_{H_j}
Probability of transmitting HIV by AIDS individual during sexual encounter	β_{A_j}
Per capita growth rate at sexual maturity age	Q_j
Average number of partners of unaware infectives	c_{I_1j}
Average number of partners of aware infectives	c_{I_2j}
Average number of partners of infectives on ART	c_{H_j}
Average number of sexual partners of AIDS persons	c_{A_j}
Probability of unaware infective to have sexual encounter with a susceptible of opposite sex	b_{I_1j}
Probability of aware infective to have sexual encounter with a susceptible of opposite sex	b_{I_2j}
Probability of infective on ART to have sexual encounter with a susceptible of opposite sex	b_{H_j}
Probability of index partner with AIDS to have sexual encounter with a susceptible of opposite sex	b_{A_j}
Rate at which unaware infectives are screened	θA_j
Rate at which unaware infective progress to AIDS	δ_j
Progression to AIDS by aware infective not on ART	ψ_j
Rate of treatment for screened and aware infective	π_j
Rate at which infective on ART progress to AIDS	ϵ_j
Per capita disease-free death rate	μ_j
Disease-related death rate for AIDS individuals	α_j

for any $j = f, m$ and each of $\alpha_j \neq 0$. In order to bring the disease under control, we require that the number of infectives ($I_{1j} \rightarrow 0$). However, this may be problematic if the total population size is varying. For example, if the total population grows faster than the infectives, then the number of infectives could grow abundantly although the proportion $i_j \rightarrow 0$. It is therefore essential to make use of proportions of the sub-populations.

3.3. Varying population size

Based on the system (1a)-(1j), we introduce proportions of individuals in the epidemiological classes as follows:

$$s_f = \frac{S_f}{N_f}, \quad i_{1f} = \frac{I_{1f}}{N_f}, \quad i_{2f} = \frac{I_{2f}}{N_f}, \quad h_f = \frac{H_f}{N_f}, \quad a_f = \frac{A_f}{N_f}.$$

Proportions for males are done in a similar manner. β_{mf} and β_{fm} can now be written as:

$$\beta_{I_{1m}} c_{I_{1m}} b_{I_{1m}} i_{1m} + \beta_{I_{2m}} c_{I_{2m}} b_{I_{2m}} i_{2m} + \beta_{H_m} c_{H_m} b_{H_m} h_m + \beta_{A_m} c_{A_m} b_{A_m} a_m$$

and

$$\beta_{I_{1f}} c_{I_{1f}} b_{I_{1f}} i_{1f} + \beta_{I_{2f}} c_{I_{2f}} b_{I_{2f}} i_{2f} + \beta_{H_f} c_{H_f} b_{H_f} h_f + \beta_{A_f} c_{A_f} b_{A_f} a_f,$$

respectively. The resulting system of differential equations is

$$\frac{ds_f}{dt} = Q_f - Q_f s_f - \beta_{mf} s_f + \alpha_f a_f s_f, \quad (3a)$$

$$\frac{ds_m}{dt} = Q_m - Q_m s_m - \beta_{fm} s_m + \alpha_m a_m s_m, \quad (3b)$$

$$\frac{di_{1f}}{dt} = \beta_{mf} s_f - (Q_f + \theta_f + \delta_f) i_{1f} + \alpha_f a_f i_{1f}, \quad (3c)$$

$$\frac{di_{1m}}{dt} = \beta_{fm} s_m - (Q_m + \theta_m + \delta_m) i_{1m} + \alpha_m a_m i_{1m}, \quad (3d)$$

$$\frac{di_{2f}}{dt} = \theta_f i_{1f} - (Q_f + \psi_f + \pi_f) i_{2f} + \alpha_f a_f i_{2f}, \quad (3e)$$

$$\frac{di_{2m}}{dt} = \theta_m i_{1m} - (Q_m + \psi_m + \pi_m) i_{2m} + \alpha_m a_m i_{2m}, \quad (3f)$$

$$\frac{dh_f}{dt} = \pi_f i_{2f} - (Q_f + \epsilon_f) h_f + \alpha_f h_f a_f, \quad (3g)$$

$$\frac{dh_m}{dt} = \pi_m i_{2m} - (Q_m + \epsilon_m) h_m + \alpha_m h_m a_m, \quad (3h)$$

$$\frac{da_f}{dt} = \delta_f i_{1f} + \psi_f i_{2f} + \epsilon_f h_f - (Q_f + \alpha_f) a_f + \alpha_f a_f^2, \quad (3i)$$

$$\frac{da_m}{dt} = \delta_m i_{1m} + \psi_m i_{2m} + \epsilon_m h_m - (Q_m + \alpha_m) a_m + \alpha_m a_m^2, \quad (3j)$$

where

$$\frac{dn_f}{dt} = Q_f(1 - n_f) + \alpha_f a_f(n_f - 1), \quad (4a)$$

$$\frac{dn_m}{dt} = Q_m(1 - n_m) + \alpha_m a_m(n_m - 1). \quad (4b)$$

If we assume that $n_i(0) \leq 1$, $i = f, m$, then it follows from equations (4a) and (4b) that $\frac{dn_i}{dt} \leq Q_i(1 - n_i)$. Therefore we conclude that $n_i(t) \leq 1$ for $i = f, m$ and a biologically feasible region for the model (3a-3j) is given by

$$\Omega_1 = \left\{ \mathbf{x} \left| \begin{array}{l} s_f, i_{1f}, i_{2f}, h_f, a_f \geq 0, \quad n_f \leq 1 \\ s_m, i_{1m}, i_{2m}, h_m, a_m \geq 0, \quad n_m \leq 1 \end{array} \right. \right\} \quad (5)$$

$$\mathbf{x} = (s_f, s_m, i_{1f}, i_{1m}, i_{2f}, i_{2m}, h_f, h_m, a_f, a_m)$$

and

$$n_j = s_j + i_{1j} + i_{2j} + h_j + a_j, \quad j = f, m.$$

Theorem 1. *The system (3a-3j) always has a disease free equilibrium point (DFE) given by $E^0 = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$.*

Proof. Using the substitutions $s_f = 1 - i_{1f} - i_{2f} - h_f - a_f$ and $s_m = 1 - i_{1m} - i_{2m} - h_m - a_m$, equations (3c) and (3d) become

$$\begin{aligned} \frac{di_{1f}}{dt} &= \beta_{mf}(1 - i_{1f} - i_{2f} - h_f - a_f) - (Q_f + \theta_f + \delta_f)i_{1f} \\ &+ \alpha_f a_f i_{1f}, \end{aligned} \quad (6a)$$

$$\begin{aligned} \frac{di_{1m}}{dt} &= \beta_{fm}(1 - i_{1m} - i_{2m} - h_m - a_m) - (Q_m + \theta_m + \delta_m)i_{1m} \\ &+ \alpha_m a_m i_{1m}. \end{aligned} \quad (6b)$$

Equations (6a) and (6b), together with equations (3e-3j) form a system in the $i_{1f}, i_{1m}, i_{2f}, i_{2m}, h_f, h_m, a_f, a_m$ region Ω_2 given by

$$\Omega_2 = \left\{ (i_{1f}, i_{1m}, i_{2f}, i_{2m}, h_f, h_m, a_f, a_m) \right\}$$

$$\left. \begin{aligned} i_{1f}, i_{2f}, h_f, a_f &\geq 0, i_{1f} + i_{2f} + h_f + a_f \leq 1 \\ i_{1m}, i_{2m}, h_m, a_m &\geq 0, i_{1m} + i_{2m} + h_m + a_m \leq 1 \end{aligned} \right\}$$

Noting that equilibrium points of the system are obtained by solving $\frac{ds_f}{dt} = \frac{ds_m}{dt} = \frac{di_{1f}}{dt} = \frac{di_{1m}}{dt} = \frac{di_{2f}}{dt} = \frac{di_{2m}}{dt} = \frac{dh_f}{dt} = \frac{dh_m}{dt} = \frac{da_f}{dt} = \frac{da_m}{dt} = 0$, it is clear that if $i_{1f} = i_{1m} = 0$ then $i_{2f} = i_{2m} = h_f = h_m = a_f = a_m = 0$ implying that $s_f = s_m = 1$. Therefore the DFE E^0 always exists and is unique. $\square$

3.3.1. Existence of a threshold parameter

The reproduction number, denoted $\mathcal{R}_0$ is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness (see [5]). The HIV/AIDS transmission model (3a)-(3h) comprises of nonnegative initial conditions and a system of equations of the form: $\dot{x} = f_i(x) = \mathcal{F}_i(x) - \mathcal{V}_i(x)$, $i = 1, 2, \dots, n$ where the functions and $\mathcal{V}_i = \mathcal{V}_i^- - \mathcal{V}_i^+$ satisfy assumptions $(A_1) - (A_5)$ in [28]. We use linear stability of the next generation operator method to find $\mathcal{R}_0$, the spectral radius of the matrix FV^{-1} . After some calculations, we get $\mathcal{R}_0 = \sqrt{\mathcal{R}_f \mathcal{R}_m}$. For $j = f, m$; we have

$$\begin{aligned} \mathcal{R}_j &= \frac{\beta_{I_{1j}} c_{I_{1j}} b_{I_{1j}}}{Q_j + \theta_j + \delta_j} + \frac{\beta_{I_{2j}} c_{I_{2j}} b_{I_{2j}} \theta_j}{(Q_j + \theta_j + \delta_j)(Q_j + \psi_j + \pi_j)} \\ &+ \frac{\beta_{H_j} c_{H_j} b_{H_j} \theta_j \pi_j}{(Q_j + \epsilon_j)(Q_j + \theta_j + \delta_j)(Q_j + \psi_j + \pi_j)} \\ &+ \frac{\beta_{A_j} c_{A_j} b_{A_j} (\theta_j \pi_j \epsilon_j + (Q_j + \epsilon_j)(\delta_j(Q_j + \psi_j + \pi_j) + \theta_j \psi_j))}{(Q_j + \alpha_j)(Q_j + \epsilon_j)(Q_j + \theta_j + \delta_j)(Q_j + \psi_j + \pi_j)}, \end{aligned}$$

$\mathcal{R}_0$ is the geometric mean of reproduction number for females ($\mathcal{R}_f$) and that for males ($\mathcal{R}_m$). $\mathcal{R}_0$ take the above form since, in heterosexual transmission, the cycle for infection for each sex must go through the opposite sex, that is, male-female-male and female-male-female. Each of the parameters used in the model (3a)-(3h) do appear in the expression of $\mathcal{R}_0$, showing that they have some influence on the asymptotic tendency of the total population size. The structures of $\mathcal{R}_f$ and $\mathcal{R}_m$ suggest that they represent sums of contact numbers of the 4 infective classes for females and males, respectively. Therefore $\mathcal{R}_0$ has the accurate conception that it is the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness. The first term in $\mathcal{R}_f$ is the contact number for

unaware female infective and it is the product of the contact rate $\beta_{I_{1f}}c_{I_{1f}}b_{I_{1f}}$ of unaware infectives per unit time and the average duration spent in that class adapted for female population growth of $1/(Q_f + \theta_f + \delta_f)$. The second term in R_f is the product of the contact rate $\beta_{h_f}c_{h_f}b_{h_f}$ of HIV individuals on treatment, the average infectious period spent in this class $1/(Q_f + \psi_f + \pi_f)$ and the fraction $\theta_f/(Q_f + \theta_f + \delta_f)$ of female infectives surviving the unaware class. The contact number for AIDS individual (fourth term in R_f) is the product of AIDS individuals' contact rate $\beta_{a_f}c_{a_f}b_{a_f}$, the average infectious period spent in this class adapted for female population growth $1/(Q_f + \alpha_f)$ and the fraction $\frac{(\theta_f\pi_f\epsilon_f + (Q_f + \epsilon_f)(\delta_f(Q_f + \psi_f + \pi_f) + \theta_f\psi_f))}{(Q_f + \epsilon_f)(Q_f + \theta_f + \delta_f)(Q_f + \psi_f + \pi_f)}$ surviving the three earlier infective stages (unaware, aware and treatment). It is possible to have in the class of AIDS individuals, a fraction that survives treatment, because of imperfect adherence (due to side effects, medication fatigue), drug resistance mutations and inadequate sustainable programs for accessing ART in limited resource settings (see [18],[20],[23]).

Theorem 2. *The DFE $E^0 = (1, 1, 0, 0, 0, 0, 0, 0, 0)$ is locally asymptotically stable in Ω_1 if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof. We use the Jacobian of the system (3a-3j) to establish the local stability of E_0 . The first two eigenvalues of J_{E_0} are $\lambda_1 = -Q_f < 0$, $\lambda_2 = -Q_m < 0$. The remaining eight eigenvalues are obtained from the block matrix

$$J_1 = \begin{pmatrix} -a & B_{1m} & 0 & B_{2m} & 0 & B_{3m} & 0 & B_{4m} \\ B_{1f} & -p & B_{2f} & 0 & B_{3f} & 0 & B_{4f} & 0 \\ \theta_f & 0 & -d & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_m & 0 & -q & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_f & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_m & 0 & -r & 0 & 0 \\ \delta_f & 0 & \psi_f & 0 & \epsilon_f & 0 & -h & 0 \\ 0 & \delta_m & 0 & \psi_m & 0 & \epsilon_m & 0 & -s \end{pmatrix}$$

$$B_{1m} = \beta_{I_{1m}}c_{I_{1m}}b_{I_{1m}}, \quad B_{2m} = \beta_{I_{2m}}c_{I_{2m}}b_{I_{2m}}, \quad B_{3m} = \beta_{H_m}c_{H_m}b_{H_m},$$

$$B_{4m} = \beta_{A_m}c_{A_m}b_{A_m}, \quad B_{1f} = \beta_{I_{1f}}c_{I_{1f}}b_{I_{1f}}, \quad B_{2f} = \beta_{I_{2f}}c_{I_{2f}}b_{I_{2f}},$$

$$B_{3f} = \beta_{H_f}c_{H_f}b_{H_f}, \quad B_{4f} = \beta_{A_f}c_{A_f}b_{A_f},$$

$$a = Q_f + \theta_f + \delta_f, \quad d = Q_f + \psi_f + \pi_f, \quad g = Q_f + \epsilon_f, \quad h = Q_f + \alpha_f,$$

$$p = Q_m + \theta_m + \delta_m, \quad q = Q_m + \psi_m + \pi_m, \quad r = Q_m + \epsilon_m, \quad s = Q_m + \alpha_m.$$

The trace and determinant of J_1 are given by:

$$\begin{aligned} \text{tr}(J_1) &= -(a + p + d + q + g + r + h + s) < 0, \\ \det(J_1) &= A_1 + A_9 - (A_2 + A_3 + A_4 + A_5 + A_6 + A_7) \\ &= (2A_1 + A_8 + A_9)(1 - \mathcal{R}_0^2) + \left(\frac{2A_1 + A_8 + A_9}{A_9} - 1\right) \sum_{i=1}^8 A_i, \end{aligned}$$

where

$$\begin{aligned} A_1 &= B_{1f}\theta_m(B_{2m}r + B_{3m}\pi_m)dghs + B_{1f}B_{4m}\theta_mdgh(\pi_m\epsilon_m \\ &\quad + \psi_mr) + B_{3f}B_{1m}\theta_f\pi_fhqrs + B_{4f}B_{1m}\theta_f\pi_f\epsilon_fqrs \\ A_2 &= B_{3f}B_{3m}\theta_f\theta_m\pi_f\pi_mhs + B_{4m}\theta_f\theta_m\pi_f\pi_m\epsilon_m(B_{3f}h + B_{4f}\epsilon_f) \\ &\quad + B_{2m}\theta_f\theta_m\pi_frs(B_{3f}h + B_{4f}\epsilon_f) \\ A_3 &= B_{4m}\theta_f\theta_m\pi_f\psi_mr(B_{3f}h + B_{4f}\epsilon_f) + B_{2f}\theta_f\theta_m\pi_mgh(B_{3m}s \\ &\quad + B_{4m}\epsilon_m) \\ A_4 &= B_{4f}\theta_f\theta_m\pi_m\psi_fg(B_{3m}s + B_{4m}\epsilon_m) + B_{2m}\theta_f\theta_mgrs(B_{2f}h \\ &\quad + B_{4f}\psi_f) + B_{4m}\theta_f\theta_m\psi_mgr(B_{2f}h + B_{4f}\psi_f) \\ A_5 &= B_{4m}\theta_f\pi_f\delta_mrq(B_{3f}h + B_{4f}\epsilon_f) + B_{1m}\theta_fgrsq(B_{2f}h \\ &\quad + B_{4f}\psi_f) + B_{4m}\theta_f\delta_mgrq(B_{2f}h + B_{4f}\psi_f) \\ A_6 &= B_{4f}\theta_m\delta_f\pi_mgd(B_{3m}s + B_{4m}\epsilon_m) + B_{4f}\theta_m\delta_fgrd(B_{2m}s \\ &\quad + B_{4m}\psi_m) + B_{1m}B_{4f}\delta_fgrdqs \\ A_7 &= B_{4f}B_{4m}\delta_f\delta_mgrdq + B_{1f}grhdq(B_{1m}s + B_{4m}\delta_m) \\ A_8 &= B_{4f}B_{3m}\theta_f\theta_m\pi_f\pi_m\epsilon_fs \\ A_9 &= adghpqrs \end{aligned}$$

so that $\mathcal{R}_0^2 = \frac{(A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8)}{A_9} = R_f R_m$. Since the second term in the expression $\det(J_1)$ is always positive, then $\det(J_1)$ is positive if $\mathcal{R}_0 < 1$. Therefore, $\lambda_i < 0$ for $i = 3, 4, \dots, 10$. Thus E_0 is locally asymptotically stable if and only if $\mathcal{R}_0 < 1$. If $\mathcal{R}_0 > 1$, then the first term of $\det(J_1)$ is positive and $\det(J_1)$ is positive since its second is always positive. Therefore, the DFE is unstable if $\mathcal{R}_0 > 1$. $\square$

Theorem 3. *If $\mathcal{R}_0 < 1$, then the DFE E^0 is globally asymptotically stable (GAS) in Ω_1 and is unstable if $\mathcal{R}_0 > 1$.*

Proof. The global asymptotic stability of the DFE is established by the existence of a Lyapunov function involving the Perron eigenvector (see [22]). According to Theorem 2.1 in [22], for the model (3a-3j) in Ω_1 , there exist a Lyapunov function given by $Q = w^T V^{-1}x$ where w is the Perron eigenvector, that is, the left eigenvector of the nonnegative matrix $V^{-1}F$ corresponding to the eigenvalue $\rho(FV^{-1}) = \mathcal{R}_0$. Here, $x = (i_{1f}, i_{1m}, i_{2f}, i_{2m}, h_f, h_m, a_f, a_m)^T$, $y = (s_f, s_m)$, and F, V, V^{-1} are calculated in Theorem 1:

$$\begin{aligned} f(x, y) &= (F - V)x - \mathcal{F}(x, y) + \mathcal{V}(x, y) \\ &= \begin{pmatrix} \beta_{mf}(1 - s_f) + \alpha_f a_f i_{1f} \\ \beta_{fm}(1 - s_m) + \alpha_m a_m i_{1mf} \\ \alpha_f a_f i_{2f} \\ \alpha_m a_m i_{2m} \\ \alpha_f a_f h_f \\ \alpha_m a_m h_m \\ \alpha_f a_f^2 \\ \alpha_m a_m^2 \end{pmatrix}. \end{aligned}$$

We note that $F \geq 0$, $V^{-1} \geq 0$, $f(x, y) \geq 0$ and let $w^T = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$ be the left eigenvector corresponding to the eigenvalue $\rho(V^{-1}F)$. Then $w^T(V^{-1}F) = (\rho(V^{-1}F))w^T$ such that $(\rho(V^{-1}F))w^T = \sqrt{\frac{B_{4m}k_f}{adgh} \frac{B_{4f}k_m}{pqrs}} w^T$, where $k_m = r(q\delta_m + \theta_m\psi_m) + \theta_m\pi_m\epsilon_m$ and $k_f = g(d\delta_f + \theta_f\psi_f) + \theta_f\pi_f\epsilon_f$. Therefore, $w^T = (0, 0, 0, 0, 0, 0, v_7, v_8)$. By Theorem 2.1 in [22], $Q = w^T V^{-1}x$ is the Lyapunov function for the system (3a)-(3j) when $\mathcal{R}_0 < 1$ and

$$\begin{aligned} Q &= \left(\frac{k_f i_{1f}}{adgh} + \frac{g\psi_f + \pi_f \epsilon_f i_{2f}}{dgh} + \frac{\epsilon_f h_f}{gh} + \frac{a_f}{h} \right) v_7 \\ &+ \left(\frac{k_m i_{1m}}{pqrs} + \frac{r\psi_m + \pi_m \epsilon_m i_{2m}}{qrs} + \frac{\epsilon_m h_m}{rs} + \frac{a_m}{s} \right) v_8. \end{aligned}$$

□

3.3.2. Existence and uniqueness of endemic equilibrium

The establishment of the endemic equilibrium is confined to the case $\mathcal{R}_0 > 1$. Letting $\mathbf{x} = (s_f, s_m, i_{1f}, i_{1m}, i_{2f}, i_{2m}, h_f, h_m, a_f, a_m)$, the endemic equilibrium point $E_1 : \mathbf{x} = \mathbf{x}^* \in \Omega_1$, is obtained by equating to zero the right hand sides

of the system (3a-3j) together with $\mathbf{x}^* > 0$. Adding the right hand sides of the equations (3a),(3c),(3e),(3g),(3i) leads to

$$Q_f(1 - s_f^* - i_{1f}^* - i_{2f}^* - h_f^* - a_f^*) - \alpha_f a_f^*(1 - s_f^* - i_{1f}^* - i_{2f}^* - h_f^* - a_f^*) = 0.$$

Therefore, $0 < a_f^* < \min\{Q_f/\alpha_f, 1\}$ and $0 < a_m^* < \min\{Q_m/\alpha_m, 1\}$. The coordinates of the endemic equilibrium E_1 as are given by:

$$i_{1f}^* = \frac{Y_f}{\theta_f \pi_f a_f^* (h - \alpha_f a_f^*)} \left(1 - a_f^* - \frac{Y_f}{Z_f} \left(1 + \frac{\theta_f}{(d - \alpha_f a_f^*)} + \frac{Z_m}{X_m} (a - \alpha_f a_f^*) \right) \right), \quad (7a)$$

$$i_{1m}^* = \frac{Y_m}{\theta_m \pi_m a_m^* (s - \alpha_m a_m^*)} \left(1 - a_m^* - \frac{Y_m}{Z_m} \left(1 + \frac{\theta_m}{(q - \alpha_m a_m^*)} + \frac{Z_f}{X_f} (p - \alpha_m a_m^*) \right) \right), \quad (7b)$$

$$i_{2f}^* = \frac{g - \alpha_f a_f^*}{\pi_f} \left(1 - a_f^* - \frac{Y_f}{Z_f} \left(1 + \frac{\theta_f}{(d - \alpha_f a_f^*)} + \frac{Z_m}{X_m} (a - \alpha_f a_f^*) \right) \right), \quad (7c)$$

$$i_{2m}^* = \frac{r - \alpha_m a_m^*}{\pi_m} \left(1 - a_m^* - \frac{Y_m}{Z_m} \left(1 + \frac{\theta_m}{(q - \alpha_m a_m^*)} + \frac{Z_f}{X_f} (p - \alpha_m a_m^*) \right) \right), \quad (7d)$$

$$h_f^* = 1 - a_f^* - \frac{Y_f}{Z_f} \left(1 + \frac{\theta_f}{(d - \alpha_f a_f^*)} + \frac{Z_m}{X_m} (a - \alpha_f a_f^*) \right), \quad (7e)$$

$$h_m^* = 1 - a_m^* - \frac{Y_m}{Z_m} \left(1 + \frac{\theta_m}{(q - \alpha_m a_m^*)} + \frac{Z_f}{X_f} (p - \alpha_m a_m^*) \right), \quad (7f)$$

with $s_f^* = 1 - i_{1f}^* - i_{2f}^* - h_f^* - a_f^*$, $s_m^* = 1 - i_{1m}^* - i_{2m}^* - h_m^* - a_m^*$,

$$X_f = B_{1f} Y_f + B_{2f} \frac{\theta_f Y_f}{d - \alpha_f a_f^*} + B_{3f} \theta_f \pi_f a_f^* (h - \alpha_f a_f^*) + B_{4f} a_f^* Z_f,$$

$$X_m = B_{1m} Y_m + B_{2m} \frac{\theta_m Y_m}{q - \alpha_m a_m^*} + B_{3m} \theta_m \pi_m a_m^* (s - \alpha_m a_m^*) + B_{4m} a_m^* Z_m,$$

$$Y_f = a_f^* (g - \alpha_f a_f^*) (h - \alpha_f a_f^*) (d - \alpha_f a_f^*),$$

$$Y_m = a_m^* (r - \alpha_m a_m^*) (s - \alpha_m a_m^*) (q - \alpha_m a_m^*),$$

$$Z_f = \theta_f \pi_f (\delta_f (d - \alpha_f a_f^*) (g - \alpha_f a_f^*) + \theta_f \psi_f (g - \alpha_f a_f^*) + \theta_f \pi_f \epsilon_f),$$

$$Z_m = \theta_m \pi_m (\delta_m (q - \alpha_m a_m^*) (r - \alpha_m a_m^*) + \theta_m \psi_m (r - \alpha_m a_m^*) + \theta_m \pi_m \epsilon_m).$$

Theorem 4. *If $\mathcal{R}_0 > 1$, then the system (3a-3j) has a unique endemic equilibrium E_1 in the interior of Ω_1 and E_1 is asymptotically stable.*

Using the Jacobian of the system (3a-3j) at the endemic equilibrium E_1 , the proof of this theorem can be easily established following the method in Theorem 2 of this paper.

Theorem 5. *If $R_0 > 1$, then the endemic equilibrium point, E_1 , of the system (3a-3j) is unique and globally asymptotically stable (GAS) in the interior of Ω_1 .*

Proof. The proof is based on the construction of a Lyapunov function (see [22]). Given the endemic equilibrium $E_1 : \mathbf{x} = \mathbf{x}^*$, we let $D_i = x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}$ where x_i represents each of the state variables of the system (3a-3j). The aim is to obtain a Lyapunov function of the form $D = \sum_{i=1}^n c_i D_i$ where $D' \leq 0$. We let $\mathbf{x}^* \equiv (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$. Differentiating along the solutions, using (3a-3j) and the inequality $1 - x + \ln x \leq 0$, $x > 0$, with equality holding if and only if $x = 1$ we get:

$$\begin{aligned}
 D'_1 &\leq B_{1m} i_{1m}^* s_f^* \left(\frac{i_{1m}}{i_{1m}^*} - \ln \frac{i_{1m}}{i_{1m}^*} - \frac{s_f}{s_f^*} + \ln \frac{s_f}{s_f^*} \right) + B_{2m} i_{2m}^* s_f^* \left(\frac{i_{2m}}{i_{2m}^*} \right. \\
 &\quad \left. - \ln \frac{i_{2m}}{i_{2m}^*} - \frac{s_f}{s_f^*} + \ln \frac{s_f}{s_f^*} \right) + B_{3m} h_m^* s_f^* \left(\frac{h_m}{h_m^*} - \ln \frac{h_m}{h_m^*} - \frac{s_f}{s_f^*} \right. \\
 &\quad \left. + \ln \frac{s_f}{s_f^*} \right) + B_{4m} a_m^* s_f^* \left(\frac{a_m}{a_m^*} - \ln \frac{a_m}{a_m^*} - \frac{s_f}{s_f^*} + \ln \frac{s_f}{s_f^*} \right) \\
 &\quad + \alpha_f a_f s_f \left(\frac{a_f^* s_f^{*2}}{a_f s_f^2} - \ln \frac{a_f^* s_f^{*2}}{a_f s_f^2} - \frac{s_f}{s_f^*} + \ln \frac{s_f}{s_f^*} \right) \\
 &= a_{14} G_{14} + a_{16} G_{16} + a_{18} G_{18} + a_{110} G_{110} + a_{19} G_{19} \\
 D'_2 &\leq B_{1f} i_{1f}^* s_m^* \left(\frac{i_{1f}}{i_{1f}^*} - \ln \frac{i_{1f}}{i_{1f}^*} - \frac{s_m}{s_m^*} + \ln \frac{s_m}{s_m^*} \right) + B_{2f} i_{2f}^* s_m^* \left(\frac{i_{2f}}{i_{2f}^*} \right. \\
 &\quad \left. - \ln \frac{i_{2f}}{i_{2f}^*} - \frac{s_m}{s_m^*} + \ln \frac{s_m}{s_m^*} \right) + B_{3f} h_f^* s_m^* \left(\frac{h_f}{h_f^*} - \ln \frac{h_f}{h_f^*} - \frac{s_m}{s_m^*} \right. \\
 &\quad \left. + \ln \frac{s_m}{s_m^*} \right) + B_{4f} a_f^* s_m^* \left(\frac{a_f}{a_f^*} - \ln \frac{a_f}{a_f^*} - \frac{s_m}{s_m^*} + \ln \frac{s_m}{s_m^*} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \alpha_m a_m s_m \left(\frac{a_m^* s_m^{*2}}{a_m s_m^2} - \ln \frac{a_m^* s_m^{*2}}{a_m s_m^2} - \frac{s_m^*}{s_m} + \ln \frac{s_m^*}{s_m} \right) \\
& = a_{23} G_{23} + a_{25} G_{25} + a_{27} G_{27} + a_{29} G_{29} + a_{210} G_{210} \\
D'_3 & \leq B_{1m} i_{1m} s_f \left(\frac{i_{1f}^* i_{1m}^* s_f^*}{i_{1f} i_{1m} s_f} - \ln \frac{i_{1f}^* i_{1m}^* s_f^*}{i_{1f} i_{1m} s_f} - \frac{i_{1f}^*}{i_{1f}} + \ln \frac{i_{1f}^*}{i_{1f}} \right) \\
& + B_{2m} i_{2m} s_f \left(\frac{i_{1f}^* i_{2m}^* s_f^*}{i_{1f} i_{2m} s_f} - \ln \frac{i_{1f}^* i_{2m}^* s_f^*}{i_{1f} i_{2m} s_f} - \frac{i_{1f}^*}{i_{1f}} + \ln \frac{i_{1f}^*}{i_{1f}} \right) \\
& + B_{3m} h_m s_f \left(\frac{i_{1f}^* h_m^* s_f^*}{i_{1f} h_m s_f} - \ln \frac{i_{1f}^* h_m^* s_f^*}{i_{1f} h_m s_f} - \frac{i_{1f}^*}{i_{1f}} + \ln \frac{i_{1f}^*}{i_{1f}} \right) \\
& + B_{4m} a_m s_f \left(\frac{i_{1f}^* a_m^* s_f^*}{i_{1f} a_m s_f} - \ln \frac{i_{1f}^* a_m^* s_f^*}{i_{1f} a_m s_f} - \frac{i_{1f}^*}{i_{1f}} + \ln \frac{i_{1f}^*}{i_{1f}} \right) \\
& + \alpha_f a_f i_{1f} \left(\frac{a_f^* i_{1f}^{*2}}{a_f i_{1f}^2} - \ln \frac{a_f^* i_{1f}^{*2}}{a_f i_{1f}^2} - \frac{i_{1f}^*}{i_{1f}} + \ln \frac{i_{1f}^*}{i_{1f}} \right) \\
& = a_{34} G_{34} + a_{36} G_{36} + a_{38} G_{38} + a_{310} G_{310} + a_{39} G_{39}
\end{aligned}$$

$$\begin{aligned}
D'_4 & \leq B_{1f} i_{1f} s_m \left(\frac{i_{1f}^* i_{1m}^* s_m^*}{i_{1f} i_{1m} s_m} - \ln \frac{i_{1f}^* i_{1m}^* s_m^*}{i_{1f} i_{1m} s_m} - \frac{i_{1m}^*}{i_{1m}} + \ln \frac{i_{1m}^*}{i_{1m}} \right) \\
& + B_{2f} i_{2f} s_m \left(\frac{i_{2f}^* i_{1m}^* s_m^*}{i_{2f} i_{1m} s_m} - \ln \frac{i_{2f}^* i_{1m}^* s_m^*}{i_{2f} i_{1m} s_m} - \frac{i_{1m}^*}{i_{1m}} + \ln \frac{i_{1m}^*}{i_{1m}} \right) \\
& + B_{3f} h_f s_m \left(\frac{h_f^* i_{1m}^* s_m^*}{h_f i_{1m} s_m} - \ln \frac{h_f^* i_{1m}^* s_m^*}{h_f i_{1m} s_m} - \frac{i_{1m}^*}{i_{1m}} + \ln \frac{i_{1m}^*}{i_{1m}} \right) \\
& + B_{4f} a_f s_m \left(\frac{a_f^* i_{1m}^* s_m^*}{a_f i_{1m} s_m} - \ln \frac{a_f^* i_{1m}^* s_m^*}{a_f i_{1m} s_m} - \frac{i_{1m}^*}{i_{1m}} + \ln \frac{i_{1m}^*}{i_{1m}} \right) \\
& + \alpha_m a_m i_{1m} \left(\frac{a_m^* i_{1m}^{*2}}{a_m i_{1m}^2} - \ln \frac{a_m^* i_{1m}^{*2}}{a_m i_{1m}^2} - \frac{i_{1m}^*}{i_{1m}} + \ln \frac{i_{1m}^*}{i_{1m}} \right) \\
& = a_{43} G_{43} + a_{45} G_{45} + a_{47} G_{47} + a_{49} G_{49} + a_{410} G_{410} \\
D'_5 & \leq \theta_f i_{1f} \left(\frac{i_{1f}^* i_{2f}^*}{i_{1f} i_{2f}} - \ln \frac{i_{1f}^* i_{2f}^*}{i_{1f} i_{2f}} - \frac{i_{2f}^*}{i_{2f}} + \ln \frac{i_{2f}^*}{i_{2f}} \right) + \alpha_f a_f i_{2f} \left(\frac{a_f^* i_{2f}^{*2}}{a_f i_{2f}^2} \right. \\
& \quad \left. - \ln \frac{a_f^* i_{2f}^{*2}}{a_f i_{2f}^2} - \frac{i_{2f}^*}{i_{2f}} + \ln \frac{i_{2f}^*}{i_{2f}} \right) = a_{53} G_{53} + a_{59} G_{59} \\
D'_6 & \leq \theta_m i_{1m} \left(\frac{i_{1m}^* i_{2m}^*}{i_{1m} i_{2m}} - \ln \frac{i_{1m}^* i_{2m}^*}{i_{1m} i_{2m}} - \frac{i_{2m}^*}{i_{2m}} + \ln \frac{i_{2m}^*}{i_{2m}} \right) + \alpha_m a_m i_{2m}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{a_m^* i_{2m}^{*2}}{a_m i_{2m}^2} \ln \frac{a_m^* i_{2m}^{*2}}{a_m i_{2m}^2} - \frac{i_{2m}^*}{i_{2m}} + \ln \frac{i_{2m}^*}{i_{2m}} \right) = a_{64} G_{64} + a_{610} G_{610} \\
D'_7 & \leq \pi_f i_{2f} \left(\frac{i_{2f}^* h_f^*}{i_{2f} h_f} - \ln \frac{i_{2f}^* h_f^*}{i_{2f} h_f} - \frac{h_f^*}{h_f} + \ln \frac{h_f^*}{h_f} \right) + \alpha_f a_f h_f \left(\frac{a_f^* h_f^{*2}}{a_f h_f^2} \right. \\
& \quad \left. - \ln \frac{a_f^* h_f^{*2}}{a_f h_f^2} - \frac{h_f^*}{h_f} + \ln \frac{h_f^*}{h_f} \right) = a_{75} G_{75} + a_{79} G_{79} \\
D'_8 & \leq \pi_m i_{2m} \left(\frac{i_{2m}^* h_m^*}{i_{2m} h_m} - \ln \frac{i_{2m}^* h_m^*}{i_{2m} h_m} - \frac{h_m^*}{h_m} + \ln \frac{h_m^*}{h_m} \right) + \alpha_m a_m h_m \\
& \quad \left(\frac{a_m^* h_m^{*2}}{a_m h_m^2} \ln \frac{a_m^* h_m^{*2}}{a_m h_m^2} - \frac{h_m^*}{h_m} + \ln \frac{h_m^*}{h_m} \right) = a_{86} G_{86} + a_{810} G_{810} \\
D'_9 & \leq \delta_f i_{1f} \left(\frac{i_{1f}^* a_f^*}{i_{1f} a_f} - \ln \frac{i_{1f}^* a_f^*}{i_{1f} a_f} - \frac{a_f^*}{a_f} + \ln \frac{a_f^*}{a_f} \right) + \psi_f i_{2f} \left(\frac{i_{2f}^* a_f^*}{i_{2f} a_f} \right. \\
& \quad \left. - \ln \frac{i_{2f}^* a_f^*}{i_{2f} a_f} - \frac{a_f^*}{a_f} + \ln \frac{a_f^*}{a_f} \right) + \epsilon_f h_f \left(\frac{h_f^* a_f^*}{h_f a_f} - \ln \frac{h_f^* a_f^*}{h_f a_f} - \frac{a_f^*}{a_f} \right. \\
& \quad \left. + \ln \frac{a_f^*}{a_f} \right) + a_f^2 \left(\frac{a_f^{*3}}{a_f^3} - \ln \frac{a_f^{*3}}{a_f^3} - \frac{a_f^*}{a_f} + \ln \frac{a_f^*}{a_f} \right) \\
& = a_{93} G_{93} + a_{95} G_{95} + a_{97} G_{97} + a_{99} G_{99} \\
D'_{10} & \leq \delta_m i_{1m} \left(\frac{i_{1m}^* a_m^*}{i_{1m} a_m} - \ln \frac{i_{1m}^* a_m^*}{i_{1m} a_m} - \frac{a_m^*}{a_m} + \ln \frac{a_m^*}{a_m} \right) \\
& \quad + \psi_m i_{2m} \left(\frac{i_{2m}^* a_m^*}{i_{2m} a_m} - \ln \frac{i_{2m}^* a_m^*}{i_{2m} a_m} - \frac{a_m^*}{a_m} + \ln \frac{a_m^*}{a_m} \right) \epsilon_m h_m \left(\frac{h_m^* a_m^*}{h_m a_m} \right. \\
& \quad \left. - \ln \frac{h_m^* a_m^*}{h_m a_m} - \frac{a_m^*}{a_m} + \ln \frac{a_m^*}{a_m} \right) + a_m^2 \left(\frac{a_m^{*3}}{a_m^3} \ln \frac{a_m^{*3}}{a_m^3} - \frac{a_m^*}{a_m} + \ln \frac{a_m^*}{a_m} \right) \\
& = a_{104} G_{104} + a_{106} G_{106} + a_{108} G_{108} + a_{1010} G_{1010}.
\end{aligned}$$

All other $a_{ij} = 0$, and we ignore loops. Recalling the results in [22], values for c_i , $i = 1, \dots, 10$ are obtained by from Kirchhoff's theorem:

$$\begin{aligned}
c_1 & = a_{16}[a_{86} + a_{810}(a_{104} + a_{106})](a_{23} + a_{25} + a_{27} + a_{29} \\
& \quad + a_{210})(a_{53}a_{64}a_{75}a_{47}) \\
c_2 & = a_{38}[a_{75} + a_{79}(a_{93} + a_{97})](a_{14} + a_{16} + a_{18} + a_{19} \\
& \quad + a_{110})(a_{49}a_{86}a_{610}a_{38}) \\
c_3 & = a_{210}a_{64}a_{86}a_{108}(a_{14} + a_{16} + a_{18} + a_{19} + a_{110})(a_{23} + a_{25} + a_{27}
\end{aligned}$$

$$\begin{aligned}
& + a_{29} + a_{210}) \\
c_4 &= a_{19}a_{75}a_{95}a_{97}(a_{34} + a_{36} + a_{38} + a_{39} + a_{310})(a_{43} + a_{45} + a_{47} \\
& + a_{49} + a_{410}) \\
c_5 &= a_{86}(a_{16} + a_{18} + a_{110} + a_{108}) + a_{106}(a_{16} + a_{810} + a_{18} + a_{110}) \\
& + a_{16}a_{810}a_{104} \\
c_6 &= a_{19}a_{53}(a_{75}a_{97} + a_{95})(a_{59} + a_{75} + a_{97} + a_{95}) + a_{27}(a_{53} + a_{75} \\
& + a_{59} + a_{93}) \\
c_7 &= a_{64}a_{104}(a_{18} + a_{610} + a_{108} + a_{106})(a_{38} + a_{86}a_{310}) \\
& + a_{59}a_{43}(a_{49} + a_{95}) \\
c_8 &= (a_{45} + a_{25} + a_{29} + a_{75})(a_{49}a_{97} + a_{64}) + a_{36}(a_{64}a_{106} + a_{104}) \\
c_9 &= a_{53}a_{75}a_{47} + (a_{34} + a_{810})(a_{610} + a_{108} + a_{86} + a_{18}) + a_{14}a_{86} \\
c_{10} &= a_{25}(a_{59}a_{93} + a_{75}a_{79}) + (a_{75} + a_{27} + a_{95} + a_{53})(a_{47}a_{79}).
\end{aligned}$$

Along each cycle: $G_{97} + G_{79} = 0$, $G_{75} + G_{97} + G_{59} = 0$,
 $G_{108} + G_{810} = 0$, $G_{86} + G_{108} + G_{610} = 0$, $G_{104} + G_{410} = 0$,
 $G_{64} + G_{86} + G_{108} + G_{410} = 0$, $G_{53} + G_{45} + G_{43} = 0$,
 $G_{95} + G_{59} = 0$, $G_{310} + G_{34} + G_{104} = 0$, $G_{49} + G_{43} + G_{93} = 0$.

Therefore, the Lyapunov function is given by $D = \sum_{i=1}^{10} c_i D_i$. D is a continuously differentiable function on the $int(\Omega_1)$ and $D(E_1) = 0$ where E_1 is the global minimum of D on Ω_1 . The time derivative of D computed along solutions of (3a)-(3j) is $D' \leq 0$ for all nonnegative population states, where equality hold only at the endemic equilibrium point E_1 and D is a Lyapunov function for system (3a)-(3j). Therefore, E_1 is a unique endemic equilibrium point of the model (3a)-(3j). Thus, E_1 is unique and globally asymptotically stable in the $int(\Omega_1)$ whenever $\mathcal{R}_0 > 1$. $\square$

4. Analysis of optimal control

We formulate an optimal control problem by introducing time dependent controls into the model (3a)-(3j). The system to be controlled is defined as

$$\begin{cases} \frac{dx}{dt} = f(x(t), u(t), t), & t_0 \leq t \leq t_{final} \\ x(t_0) = x_0, & \text{given,} \end{cases} \quad (8)$$

where the vector $x \in \mathbf{R}^n$ are the variables, that is, homogeneous sub-populations at each instant of time; $u \in \mathbf{R}^p$ are control functions of time; the vector $f(t, x, u) : [t_0 = 0, t_{final} = T] \times \mathbf{R}^n \times \mathbf{R}^p$ and $\frac{df}{dx}$ are continuous functions with respect to all their arguments; $n = 10$, $p = 8$. The control $u_{j1}(t)$ ($j = f, m$) measures public health education and influences good sexual behaviour such as sticking to one partner while $u_{j2}(t)$ represents the prevention control which reduces infectiousness of the HIV infectives. Examples of prevention methods include use of condoms, HIV prophylaxis, and voluntary male circumcision. $u_{j3}(t)$ measures the rate of screening of unaware infectives at each time period while $u_{j4}(t)$ measures the rate at which HIV infectives are treated. Each of the controls is bounded and may be a function of sex/health education and other factors. The model (3a)-(3j) becomes

$$\frac{ds_f}{dt} = Q_f - (Q_f + u_{f1})s_f - (1 - u_{f2})\beta_{mf}s_f + \alpha_f a_f s_f, \quad (9a)$$

$$\begin{aligned} \frac{ds_m}{dt} &= Q_m - (Q_m + u_{m1})s_m - (1 - u_{m2})\beta_{fm}s_m \\ &+ \alpha_m a_m s_m, \end{aligned} \quad (9b)$$

$$\begin{aligned} \frac{di_{1f}}{dt} &= (1 - u_{f2})\beta_{mf}s_f - (Q_f + \theta_f + \delta_f + u_{f3})i_{1f} \\ &+ \alpha_f a_f i_{1f}, \end{aligned} \quad (9c)$$

$$\begin{aligned} \frac{di_{1m}}{dt} &= (1 - u_{m2})\beta_{fm}s_m - (Q_m + \theta_m + \delta_m + u_{m3})i_{1m} \\ &+ \alpha_m a_m i_{1m}, \end{aligned} \quad (9d)$$

$$\begin{aligned} \frac{di_{2f}}{dt} &= (\theta_f + u_{f3})i_{1f} - (Q_f + \psi_f + \pi_f + u_{f4})i_{2f} \\ &+ \alpha_f a_f i_{2f}, \end{aligned} \quad (9e)$$

$$\begin{aligned} \frac{di_{2m}}{dt} &= (\theta_m + u_{m3})i_{1m} - (Q_m + \psi_m + \pi_m + u_{m4})i_{2m} \\ &+ \alpha_m a_m i_{2m}, \end{aligned} \quad (9f)$$

$$\frac{dh_f}{dt} = (\pi_f + u_{f4})i_{2f} - (Q_f + \epsilon_f)h_f + \alpha_f h_f a_f, \quad (9g)$$

$$\frac{dh_m}{dt} = (\pi_m + u_{m4})i_{2m} - (Q_m + \epsilon_m)h_m + \alpha_m h_m a_m, \quad (9h)$$

$$\frac{da_f}{dt} = \delta_f i_{1f} + \psi_f i_{2f} + \epsilon_f h_f - (Q_f + \alpha_f)a_f + \alpha_f a_f^2, \quad (9i)$$

$$\frac{da_m}{dt} = \delta_m i_{1m} + \psi_m i_{2m} + \epsilon_m h_m - (Q_m + \alpha_m)a_m + \alpha_m a_m^2, \quad (9j)$$

with non-negative known initial conditions $\mathbf{x}(0)$ and controls $0 \leq u_{jk} \leq 1$ for $j = f, m; k = 1, 2, 3, 4$. Our goal is to seek optimal levels of the intervention strategies required to minimise the proportion of the susceptibles, unaware infectives, and the cost of applying the controls. We choose the objective functional J defined as:

$$\begin{aligned} J = & \int_0^T [a_1 s_f + a_2 s_m + a_3 i_{1f} + a_4 i_{1m} + b_1 u_{f1}^2 + b_2 u_{m1}^2 \\ & + b_3 u_{f2}^2 + b_4 u_{m2}^2 + b_5 u_{f3}^2 + b_6 u_{m3}^2 + b_7 u_{f4}^2 + b_8 u_{m4}^2] dt \end{aligned} \quad (10)$$

with a_1, a_2, a_3, a_4 being positive weight constants of the susceptibles and unaware infectives. The quantities $b_k, k = 1, \dots, 8$ are weight constants for the respective controls. The terms $b_k u_{jk}, j = f, m; k = 1, 2, 3, 4$ describe the costs associated with public health education, prevention control (e.g., distribution of condoms), screening of unaware infectives and treatment of infectives. A quadratic functional is chosen by assuming nonlinear costs emanating from the control measures. We define the class of admissible controls by $U = \{\mathbf{u} \in L^1(0, T) | 0 \leq u_{jk}(t) \leq 1 \forall t \in [0, T]\}$ where $\mathbf{u} = (u_{f1}, u_{m1}, u_{f2}, u_{m2}, u_{f3}, u_{m3}, u_{f4}, u_{m4})$. U is assumed to consist of piecewise continuous controls. Our aim is to determine an optimal control set $\mathbf{u}^*$:

$$J(\mathbf{u}^*) = \min_U \{J(\mathbf{u})\}. \quad (11)$$

4.1. Existence of an optimal control set

Theorem 6. *There exists an optimal control $\mathbf{u}^*$ and corresponding solution $\mathbf{x}^*$ to the system (9a) – (9j): $J(\mathbf{u}^*) = \min_U \{J(\mathbf{u})\}$.*

Proof. We make reference to Theorem 4.1 and its accompanying Corollary 4.1 in Chapter III of [7]. We give the theorem's conditions below and later prove them:

1. The set of all solutions to the system (9a)-(9j) with corresponding controls in U is non-empty;
2. The control set U is closed;
3. The state system (9a)-(9j) is linear in the control variables with coefficients depending on state variables and time;

4. The integrand L of the objective functional J is convex in the controls;
5. $L(t, \mathbf{x}, \mathbf{u}) \geq c_1|\mathbf{u}|^\beta - c_2$ where $c_1 > 0$ and $\beta > 1$.

In order to verify condition 1, we use Picard-Lindelöf Theorem 3.1 on page 12 of [4]. If the state equations (9a)-(9j) are continuous and Lipschitz in the state variables and the corresponding solutions to the state equations are bounded, then there exist a unique solution corresponding to every admissible control in U . The total population proportions n_j and their corresponding sub-populations for each sex are bounded below by zero and above by 1. Therefore, the state system (9a)-(9j) is continuous and bounded. In order to guarantee uniqueness of the solution, we establish that the system (9a)-(9j) is Lipschitz regarding the state variables. This can be straightly obtained by showing that partial derivatives in the system (9a)-(9j) with respect to the state variables exist and are bounded. Thus, condition 1 is proved. Clearly, our control set U satisfies condition 2. Condition 3 is proved by noticing that the RHS of the state system linearly depend on the controls. The integrand L of J is evidently convex on U since L is quadratic in the controls. Therefore, condition 4 is proved. To prove condition 3, we note that

$$\begin{aligned}
 L &= a_1 s_f + a_2 s_m + a_3 i_{1f} + a_4 i_{1m} + b_1 u_{f1}^2 + b_2 u_{m1}^2 + b_3 u_{f2}^2 \\
 &+ b_4 u_{m2}^2 + b_5 u_{f3}^2 + b_6 u_{m3}^2 + b_7 u_{f4}^2 + b_8 u_{m4}^2 \\
 &\geq b_2 u_{m1}^2 + b_3 u_{f2}^2 + b_4 u_{m2}^2 + b_5 u_{f3}^2 + b_6 u_{m3}^2 + b_7 u_{f4}^2 + b_8 u_{m4}^2 \\
 &\geq b_1 u_{f1}^2 + b_2 u_{m1}^2 + b_3 u_{f2}^2 + b_4 u_{m2}^2 + b_5 u_{f3}^2 + b_6 u_{m3}^2 + b_7 u_{f4}^2 \\
 &+ b_8 u_{m4}^2 - b_1 \\
 &\geq \min(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)(u_{f1}^2 + u_{m1}^2 + u_{f2}^2 + u_{m2}^2 + u_{f3}^2 \\
 &+ u_{m3}^2 + u_{f4}^2 + u_{m4}^2) - b_1
 \end{aligned}$$

since $0 \leq u_{f1} \leq 1$ implies that $(b_1 u_{f1}^2 - b_1) \leq 0$. Therefore, there exists an optimal control pair $(\mathbf{x}^*, \mathbf{u}^*)$ which solves the optimal control problem (that is, minimise (10) subject to (9a) – (9j)). $\square$

4.2. Characterisation of the optimal controls

We apply Pontryagin's Maximum Principle (see [21]) to derive the necessary conditions for the optimal control pair, $(\mathbf{x}, \mathbf{u})$, whose existence was proved in Section 4.1. The system (9a)-(9j) and (11) are converted to a problem of minimising pointwise a non-linear C^2 function called the Hamiltonian $H(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{\Lambda}(t))$,

with respect to $\mathbf{u}$ where

$$\mathbf{\Lambda} = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t), \lambda_7(t), \lambda_8(t), \lambda_9(t), \lambda_{10}(t))^T.$$

Theorem 7. *Given the optimal control pair $(\mathbf{x}^*, \mathbf{u}^*)$ that minimises the objective functional (10) subject to (9a) – (9j), there exists a piecewise differentiable adjoint variable $\mathbf{\Lambda}(t)$, satisfying*

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -a_1 + (1 - u_{f2}^*)\beta_{mf}^*(\lambda_1 - \lambda_3) + (Q_f + u_{f1}^* - \alpha_f a_f^*)\lambda_1 \\ \frac{d\lambda_2}{dt} &= -a_2 + (1 - u_{m2}^*)\beta_{fm}^*(\lambda_2 - \lambda_4) + (Q_m + u_{m1}^* - \alpha_m a_m^*)\lambda_2 \\ \frac{d\lambda_3}{dt} &= -a_3 + (1 - u_{m2}^*)s_m^* B_{1f}^*(\lambda_2 - \lambda_4) + (\theta_f + u_{f3}^*)(\lambda_3 - \lambda_5) \\ &\quad + \delta_f(\lambda_3 - \lambda_9) + (Q_f - \alpha_f a_f^*)\lambda_3 \\ \frac{d\lambda_4}{dt} &= -a_4 + (1 - u_{f2}^*)s_f^* B_{1m}^*(\lambda_3 - \lambda_5) + (\theta_m + u_{m3}^*)(\lambda_4 - \lambda_6) \\ &\quad + \delta_m(\lambda_4 - \lambda_{10}) + (Q_m - \alpha_m a_m^*)\lambda_4 \\ \frac{d\lambda_5}{dt} &= (1 - u_{m2}^*)s_m^* B_{2f}^*(\lambda_2 - \lambda_4) + (\pi_f + u_{f4}^*)(\lambda_5 - \lambda_7) \\ &\quad + \psi_f(\lambda_5 - \lambda_9) + (Q_f - \alpha_f a_f^*)\lambda_5 \\ \frac{d\lambda_6}{dt} &= (1 - u_{f2}^*)s_f^* B_{2m}^*(\lambda_3 - \lambda_5) + (\pi_m + u_{m4}^*)(\lambda_6 - \lambda_8) \\ &\quad + \psi_m(\lambda_6 - \lambda_{10}) + (Q_m - \alpha_m a_m^*)\lambda_6 \\ \frac{d\lambda_7}{dt} &= (1 - u_{m2}^*)s_m^* B_{3f}^*(\lambda_2 - \lambda_4) + \epsilon_f(\lambda_7 - \lambda_9) \\ &\quad + (Q_f - \alpha_f a_f^*)\lambda_7 \\ \frac{d\lambda_8}{dt} &= (1 - u_{f2}^*)s_f^* B_{3m}^*(\lambda_3 - \lambda_5) + \epsilon_m(\lambda_8 - \lambda_{10}) \\ &\quad + (Q_m - \alpha_m a_m^*)\lambda_8 \\ \\ \frac{d\lambda_9}{dt} &= -\alpha_f(s_f^*\lambda_1 + i_{1f}^*\lambda_3 + i_{2f}^*\lambda_5 + h_f^*\lambda_7) \\ &\quad + (1 - u_{m2}^*)s_m^* B_{4f}^*(\lambda_2 - \lambda_4) + (Q_f + \alpha_f - 2\alpha_f a_f^*)\lambda_9 \\ \frac{d\lambda_{10}}{dt} &= -\alpha_m(s_m^*\lambda_2 + i_{1m}^*\lambda_4 + i_{2m}^*\lambda_6 + h_m^*\lambda_8) \\ &\quad + (1 - u_{f2}^*)s_f^* B_{4m}^*(\lambda_3 - \lambda_5) + (Q_m + \alpha_m - 2\alpha_m a_m^*)\lambda_{10} \end{aligned}$$

with transversality conditions $\lambda_i(T) = 0 \ \forall i = 1, \dots, 10$. Moreover, the optimal

control set is characterised by the continuous functions

$$\begin{aligned}
 u_{f1}^* &= \min \left\{ 1, \max \left\{ 0, \frac{s_f^*}{2b_1} \lambda_1 \right\} \right\} \\
 u_{m1}^* &= \min \left\{ 1, \max \left\{ 0, \frac{s_m^*}{2b_2} \lambda_2 \right\} \right\} \\
 u_{f2}^* &= \min \left\{ 1, \max \left\{ 0, \frac{\beta_{mf}^* s_f^*}{2b_3} (\lambda_3 - \lambda_1) \right\} \right\} \\
 u_{m2}^* &= \min \left\{ 1, \max \left\{ 0, \frac{\beta_{fm}^* s_m^*}{2b_4} (\lambda_4 - \lambda_2) \right\} \right\} \\
 u_{f3}^* &= \min \left\{ 1, \max \left\{ 0, \frac{i_{1f}^*}{2b_5} (\lambda_3 - \lambda_5) \right\} \right\} \\
 u_{m3}^* &= \min \left\{ 1, \max \left\{ 0, \frac{i_{1m}^*}{2b_6} (\lambda_4 - \lambda_6) \right\} \right\} \\
 u_{f4}^* &= \min \left\{ 1, \max \left\{ 0, \frac{i_{2f}^*}{2b_7} (\lambda_5 - \lambda_7) \right\} \right\} \\
 u_{m4}^* &= \min \left\{ 1, \max \left\{ 0, \frac{i_{2m}^*}{2b_8} (\lambda_6 - \lambda_8) \right\} \right\}.
 \end{aligned} \tag{12}$$

Proof. We systematically apply Pontryagin's Maximum Principle (see [13],[21]) to obtain the required necessary conditions. The Hamiltonian, H , is given by

$$\begin{aligned}
 H &= a_1 s_f + a_2 s_m + a_3 i_{1f} + a_4 i_{1m} + b_1 u_{f1}^2 + b_2 u_{m1}^2 + b_3 u_{f2}^2 \\
 &+ b_4 u_{m2}^2 + b_5 u_{f3}^2 + b_6 u_{m3}^2 + b_7 u_{f4}^2 + b_8 u_{m4}^2 + \lambda_1 [Q_f - (Q_f \\
 &+ u_{f1})s_f - (1 - u_{f2})\beta_{mf}s_f + \alpha_f a_f s_f] + \lambda_2 [Q_m - (Q_m \\
 &+ u_{m1})s_m - (1 - u_{m2})\beta_{fm}s_m + \alpha_m a_m s_m] + \lambda_3 [(1 - u_{f2})\beta_{mf}s_f \\
 &- (Q_f + \theta_f + \delta_f + u_{f3})i_{1f} + \alpha_f a_f i_{1f}] + \lambda_4 [(1 - u_{m2})\beta_{fm}s_m \\
 &- (Q_m + \theta_m + \delta_m + u_{m3})i_{1m} + \alpha_m a_m i_{1m}] + \lambda_5 [(\theta_f + u_{f3})i_{1f} \\
 &- (Q_f + \psi_f + \pi_f + u_{f4})i_{2f} + \alpha_f a_f i_{2f}] + \lambda_6 [(\theta_m + u_{m3})i_{1m} \\
 &- (Q_m + \psi_m + \pi_m + u_{m4})i_{2m} + \alpha_m a_m i_{2m}] + \lambda_7 [(\pi_f + u_{f4})i_{2f} \\
 &- (Q_f + \epsilon_f)h_f + \alpha_f h_f a_f] + \lambda_8 [(\pi_m + u_{m4})i_{2m} - (Q_m + \epsilon_m)h_m \\
 &+ \alpha_m h_m a_m] + \lambda_9 [\delta_f i_{1f} + \psi_f i_{2f} + \epsilon_f h_f - (Q_f + \alpha_f)a_f + \alpha_f a_f^2] \\
 &+ \lambda_{10} [\delta_m i_{1m} + \psi_m i_{2m} + \epsilon_m h_m - (Q_m + \alpha_m)a_m + \alpha_m a_m^2].
 \end{aligned}$$

We use the equations $\frac{d\lambda_j}{dt} = -\frac{\partial H}{\partial x_j}$ to get the adjoint variables and they must

satisfy the transversality conditions $\lambda_j = 0 \ \forall i = 1, 2, \dots, 10$. We apply the optimality equation $\frac{\partial H}{\partial u_{jk}^*} = 0$ for all values $j = f, m; k = 1, 2, 3, 4$ in order to find characterisation of the optimal control $\mathbf{u}^*$ in terms of $\mathbf{x}^*$ and $\mathbf{\Lambda}$:

$$\begin{aligned} \frac{\partial H}{\partial u_{f1}^*} &= 2b_1 u_{f1}^* - s_f^* \lambda_1; & \frac{\partial H}{\partial u_{m1}^*} &= 2b_2 u_{m1}^* - s_m^* \lambda_2; \\ \frac{\partial H}{\partial u_{f2}^*} &= 2b_3 u_{f2}^* + \beta_{mf}^* s_f^* (\lambda_1 - \lambda_3); & \frac{\partial H}{\partial u_{m2}^*} &= 2b_4 u_{m2}^* + \beta_{fm}^* s_m^* (\lambda_2 - \lambda_4); \\ \frac{\partial H}{\partial u_{f3}^*} &= 2b_5 u_{f3}^* + i_{1f}^* (\lambda_5 - \lambda_3); & \frac{\partial H}{\partial u_{m3}^*} &= 2b_6 u_{m3}^* + i_{1m}^* (\lambda_6 - \lambda_4); \\ \frac{\partial H}{\partial u_{f4}^*} &= 2b_7 u_{f4}^* + i_{2f}^* (\lambda_7 - \lambda_5); & \frac{\partial H}{\partial u_{m4}^*} &= 2b_8 u_{m4}^* + i_{2m}^* (\lambda_8 - \lambda_6). \end{aligned}$$

We obtain:

$$\begin{aligned} u_{f1}^* &= \frac{s_f^*}{2b_1} \lambda_1; & u_{m1}^* &= \frac{s_m^*}{2b_2} \lambda_2; & u_{f2}^* &= \frac{\beta_{mf}^* s_f^*}{2b_3} (\lambda_3 - \lambda_1); \\ u_{m2}^* &= \frac{\beta_{fm}^* s_m^*}{2b_4} (\lambda_4 - \lambda_2); & u_{f3}^* &= \frac{i_{1f}^*}{2b_5} (\lambda_3 - \lambda_5); & u_{m3}^* &= \frac{i_{1m}^*}{2b_6} (\lambda_4 - \lambda_6); \\ u_{f4}^* &= \frac{i_{2f}^*}{2b_7} (\lambda_5 - \lambda_7); & u_{m4}^* &= \frac{i_{2m}^*}{2b_8} (\lambda_6 - \lambda_8). \end{aligned}$$

Applying the bounds on the controls $u_j^* = \begin{cases} u_j & \text{if } 0 < u_j < 1 \\ 0 & \text{if } u_j \leq 0 \\ 1 & \text{if } u_j \geq 1 \end{cases}$, we obtain

the required characterisation. Therefore Theorem 7 is proved. $\square$

5. Numerical results

Numerical simulations to approximate the optimal controls are carried out using the Forward-Backward Sweep method whose algorithm is described on page 50 in [13]. The optimality system consists of twenty ordinary differential equations having 10 state equations and 10 adjoint equations. State equations have initial conditions and adjoint equations have boundary conditions at the final time T . Then simultaneously, state space variables are computed forward in time and the adjoints (or costates) are computed backward in time in the same interval together with the solution of the state variables transversality conditions. Adjoint equations are computed backward in time since transversality

conditions are final time conditions. We provide an initial guess for the controls for the process to start and the controls are continuously updated using state and adjoint values with the characterisation in (12). We adjust the initial guess in instances where convergence problems are encountered. The whole process is repeated until convergence occurs, that is, when current state, adjoint and control values are sufficiently close to successive values. We plot states in the absence of controls and in the presence of controls to determine the best intervention strategies.

We use different initial conditions and weight constants

$a_1, a_2, a_3, a_4; b_i, i = 1, \dots, 8$ to obtain several control strategies for a 50-year plan. Figures (1)-(6) were obtained using

$\mathbf{x}(0) = (0.860, 0.800, 0.057, 0.100, 0.029, 0.030, 0.036, 0.040, 0.018,$

$0.030)$ and weight constants $a_1 = 7, a_2 = 5, a_3 = 2, a_4 = 3$ and $b_1 = 14, b_2 = 12, b_3 = 28, b_4 = 24, b_5 = 6, b_6 = 7, b_7 = 52, b_8 = 48$. The weight constants are assumed to be proportional to the cost of the controls. The other parameters are shown in Table 2.

Figure 1: Variation of female population without controls

When no controls are administered, unaware infectives rise from below 10% of the population and level around 40% of the population in about two years as shown in Figures 1 and 2. However, in the presence of controls, Figure 3 shows that the unaware infectives reach their peak within one year and falls to about 10% of the population within the next four years. In the case where all controls are fully administered, the susceptible population immediately starts to decrease to about 10% of the population in the first two years

Table 2: Parameter values used: The assumed values are based on HIV/AIDS dynamics of Zimbabwe, one of the Southern African countries

Symbol	Female	Male	Source
$\beta_{I_{1i}}$	0.28	0.42	see ([1],[29])
$\beta_{I_{2i}}$	0.18	0.24	see ([1],[29])
β_{H_i}	0.14	0.16	see ([1],[29])
β_{A_i}	0.10	0.12	see ([1],[29])
Q_i	0.016	0.012	Assumed
$c_{I_{1i}}$	4	6	see ([3],[12])
$c_{I_{2i}}$	3	4	see ([3],[12])
c_{H_i}	2	3	see ([3],[12])
c_{A_i}	1	2	see ([3],[12])
$b_{I_{1i}}$	0.7	0.8	Assumed
$b_{I_{2i}}$	0.65	0.75	Assumed
b_{H_i}	0.5	0.6	Assumed
b_{A_i}	0.3	0.4	Assumed
θ_i	0.8	0.69	see ([17],[33])
δ_i	0.24	0.20	see ([31])
ψ_i	0.16	0.12	see ([1],[24],[31])
π_i	0.62	0.51	see ([17])
ϵ_i	0.0018	0.0011	see ([31])
α_j	0.025	0.021	see ([30],[33])
α_j	0.025	0.021	see ([30],[33])
μ_i	0.02	0.018	see ([17],[33])

Figure 2: Variation of male population without controls

Figure 3: Variation of population $(s_f, s_m, i_{1f}, i_{1m})$ when all controls are applied

Figure 4: Variation of population $(i_{2f}, i_{2m}, h_f, h_m)$ when all controls are applied

Figure 5: Variation of population (a_f, a_m) and optimal controls u_{f1}, u_{m1} when all controls are applied

Figure 6: Profiles for controls u_{f2} , u_{m2} , u_{f3} and u_{m3} when all controls are applied. Profiles for controls u_{f4} and u_{m4} are not shown since they are similar to those for u_{f2} and u_{m2}

and fewer females than males are susceptible (see Figure 3). Control profiles for u_{f1} , u_{m1} , u_{f2} , u_{m2} , u_{f4} and u_{m4} are similar. The sex education (u_{f1} , u_{m1}) and HIV prevention (u_{f2} , u_{m2}) controls reduce the susceptible population, while the screening control (u_{f3} , u_{m3}) decreases the unaware infective population, thereby minimising HIV transmission. Moreover, the screening control gives rise to the aware infectives which are then recruited for treatment. The combination of all controls is responsible for the reduction of AIDS individuals to almost zero, that is, the effect of appropriate education, prevention, screening and treatment.

6. Conclusion

A heterosexual HIV/AIDS model is formulated and analysed, especially its well-posedness, existence of a disease threshold parameter and asymptotic stability of disease free and endemic equilibrium points. Apart from being a variable population size model, our model assumes asymmetry in female and male HIV transmission and progression rates, a feature that is not evident in models in literature, to the best of our knowledge. Optimal control theories are applied to establish the existence of an optimal control solution and its characterisation. We numerically solve the optimal control problem (since such problems are difficult to solve analytically) using the forward-backward sweep method

based on Runge Kutta fourth order scheme. Administering in full scale, all the four controls (sex education, HIV prevention methods, screening, treatment) produces the optimal control plan in which susceptibles and unaware infectives are reduced to below 10% of the population while all screened infectives are recruited for treatment, thereby greatly reducing the epidemic. In practice, HIV/AIDS dynamics is more complicated than as depicted in this model. However, our study demonstrates the benefits of mathematical modelling in reducing the HIV/AIDS disease.

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