L^γ INEQUALITIES
CONCERNING POLYNOMIALS

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Abstract: In this paper, first, we obtain L^γ analogue of an inequality concerning a polynomial $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ of degree n having no zero in $|z| < k$, $k \geq 1$ proved by Govil et al. [*Illinois Jour. of Math.*, **23**, No 2 (1979), 319-329].

Second, we also prove L^γ analogue of another inequality for a polynomial $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ of degree n having all its zeros in $|z| \leq k$, $k \leq 1$ proved in the same paper. The results improve other known inequalities as well.

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1. Introduction and statement of results

Let $\mathbb{P}_n$ be the class of polynomials $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ of degree n . For $p \in \mathbb{P}_n$, define

$$\|p\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \quad \gamma > 0. \quad (1.1)$$

From a well known fact of analysis ([15],[16]), we know

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$$\lim_{\gamma \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} = \max_{|z|=1} |p(z)|. \quad (1.2)$$

Hence, we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|. \quad (1.3)$$

Further, if we denote $\|p\|_0 = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\}$, then a simple calculation shows that

$$\lim_{\gamma \rightarrow 0^+} \|p\|_\gamma = \|p\|_0.$$

If $p \in \mathbb{P}_n$, Bernstein [3] proved that

$$\|p'\|_\infty \leq n\|p\|_\infty. \quad (1.4)$$

Inequality (1.4) is known as Bernstein's inequality where equality occurs for $p(z) = \alpha z^n$, $\alpha \neq 0$. Inequality (1.4) can be obtained by letting $\gamma \rightarrow \infty$ in the inequality

$$\|p'\|_\gamma \leq n\|p\|_\gamma, \quad \gamma > 0, \quad (1.5)$$

which was proved by Zygmund [18] for $\gamma \geq 1$ for all trigonometric polynomials of degree n and not only for those which are of the form $p(e^{i\theta})$. The validity of (1.5) for $0 < \gamma < 1$ was proved by Arestov [1].

If $p \in \mathbb{P}_n$ such that $p(z) \neq 0$ in $|z| < 1$, then inequalities (1.4) and (1.5) can be replaced respectively by

$$\|p'\|_\infty \leq \frac{n}{2} \|p\|_\infty \quad (1.6)$$

and

$$\|p'\|_\gamma \leq \frac{n}{\|1+z\|_\gamma} \|p\|_\gamma, \quad \gamma > 0. \quad (1.7)$$

While inequality (1.6) was conjectured by Erdős and later proved by Lax [10], inequality (1.7) was proved by de Bruijn [4] for $\gamma \geq 1$. Rahman and Schmeisser [13] showed that (1.7) remain true for $0 < \gamma < 1$.

On the other hand, if $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq 1$, Turán [17] proved that

$$\|p'\|_\infty \geq \frac{n}{2} \|p\|_\infty. \quad (1.8)$$

Both inequalities (1.6) and (1.8) attain equality for the polynomial $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

Malik [11] generalized (1.6) by considering that $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k \geq 1$ and proved that

$$\|p'\|_\infty \leq \frac{n}{1+k} \|p\|_\infty. \quad (1.9)$$

(1.9) is best possible and equality holds for $p(z) = (z+k)^n$.

Under the same hypotheses for the polynomial as above, Govil and Rahman [7] extended (1.7) into L^γ version by proving that

$$\|p'\|_\gamma \leq \frac{n}{\|k+z\|_\gamma} \|p\|_\gamma, \quad \gamma \geq 1. \quad (1.10)$$

Gardner and Weems [6] and Rather [14] independently proved that (1.10) also holds for $0 < \gamma < 1$.

Govil et al. [8] refined (1.9) for $p \in \mathbb{P}_n$ such that $p(z) \neq 0$ in $|z| < k$, $k \geq 1$ by proving that

$$\|p'\|_\infty \leq n \left\{ \frac{n|a_0| + k^2|a_1|}{n|a_0|(1+k^2) + 2k^2|a_1|} \right\} \|p\|_\infty. \quad (1.11)$$

Inequality (1.11) was extended to L^γ norm by Aziz and Rather [2]. They proved that if $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k \geq 1$, then for each $\gamma > 0$

$$\|p'\|_\gamma \leq \frac{n}{\|\delta_{k,1} + z\|_\gamma} \|p\|_\gamma, \quad (1.12)$$

where $\delta_{k,1} = \frac{n|a_0|k^2 + |a_1|k^2}{n|a_0| + k^2|a_1|}$.

On the other hand, as a generalization of (1.8), Malik [11] proved that if $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k} \|p\|_\infty. \quad (1.13)$$

Malik [12] generalized (1.13) in the sense that the inequality involves integral mean of $|p(z)|$ on $|z| = 1$ and extends to L^γ norm. He proved that if $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq 1$, then for each $\gamma > 0$

$$\|p'\|_\gamma \geq \frac{n}{\|1+z\|_\gamma} \|p\|_\gamma. \quad (1.14)$$

As a generalization of 1.14, Aziz and Rather [2] proved that if $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $\gamma > 0$

$$n \left\| \frac{p}{p'} \right\|_\gamma \leq \|1 + t_{k,1}z\|_\gamma \quad (1.15)$$

and

$$\|p'\|_\infty \geq \frac{n}{\|1 + t_{k,1}z\|_\gamma} \|p\|_\gamma, \quad (1.16)$$

$$\text{where } t_{k,1} = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|}.$$

Several improvements, generalizations and extensions of the above inequalities which fundamentally estimate the bounds of $\left\{ \frac{\|p'\|_\infty}{\|p\|_\infty} \right\}$ under prescribed restrictions on zeros of $p(z)$ are available in literature. It is also desirable to know the dependence of the above mentioned ratio on the coefficients of the polynomial under consideration. In this direction Govil et al. [8] proved the following two results, where the first improves (1.9) and (1.11), and the second (1.13) by involving certain coefficients of the polynomial.

Theorem A. *If $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k \geq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \|p\|_\infty, \quad (1.17)$$

where

$$\lambda = \frac{k}{n} \frac{a_1}{a_0} \quad \text{and} \quad \mu = \frac{2k^2}{n(n-1)} \frac{a_2}{a_0}.$$

Theorem B. *If $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$\|p'\|_\infty \geq \frac{n}{1+k} \frac{(1-|\omega|)(1+k^2|\omega|) + k(n-1)|\Omega - \omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|) + k(n-1)|\Omega - \omega^2|} \|p\|_\infty, \quad (1.18)$$

where

$$\omega = \frac{1}{nk} \frac{\bar{a}_{n-1}}{\bar{a}_n} \quad \text{and} \quad \Omega = \frac{2}{n(n-1)k^2} \frac{\bar{a}_{n-2}}{\bar{a}_n}. \quad (1.19)$$

In this paper we extend both Theorem A and Theorem B to L^γ norms. Our results influence many of known inequalities including the inequalities discussed above.

First, we obtain the following result which is L^γ analogue of Theorem A. It not only generalizes Theorem A but also improves inequalities (1.10) and (1.12) directly while its ordinary version improves (1.9) and (1.11).

Theorem 1. *If $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k \geq 1$, then for each $\gamma > 0$*

$$\|p'\|_\gamma \leq \frac{n}{\|C + z\|_\gamma} \|p\|_\gamma, \quad (1.20)$$

where

$$C = \frac{k(1 - |\lambda|)(|\lambda| + k^2) + k(n - 1)|\mu - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda|k^2) + k(n - 1)|\mu - \lambda^2|}, \quad (1.21)$$

$$\lambda = \frac{k}{n} \frac{a_1}{a_0} \quad \text{and} \quad \mu = \frac{2k^2}{n(n - 1)} \frac{a_2}{a_0}.$$

Remark 1. Inequality (1.20) is an improvement of (1.10) proved by Govil and Rahman [7] for $\gamma \geq 1$ and, Gardner and Weems [6] and Rather [14] for $0 < \gamma < 1$. It is sufficient to prove that $C \geq k$, where C is defined by (1.21) in Theorem 1 and $k \geq 1$, which is equivalent to showing that

$$\frac{(1 - |\lambda|)(|\lambda| + k^2) + k(n - 1)|\mu - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda|k^2) + k(n - 1)|\mu - \lambda^2|} \geq 1,$$

$$\text{i.e. } |\lambda| + k^2 \geq 1 + |\lambda|k^2,$$

$$\text{i.e. } k^2(1 - |\lambda|) \geq 1 - |\lambda|,$$

$$\text{i.e. } k^2 \geq 1,$$

which is true since $1 - |\lambda| \geq 0$ (by Lemma 2.1) and $k \geq 1$.

Remark 2. Inequality (1.20) is also an improvement of (1.12) proved by Aziz and Rather [2]. For this, it is enough to show that $C \geq \delta_{k,1}$, where C and $\delta_{k,1}$ are defined by (1.21) in Theorem 1 and inequality (1.12) respectively.

Since $\delta_{k,1} = \frac{n|a_0|k^2 + |a_1|k^2}{n|a_0| + |a_1|k^2} = \frac{k + |\lambda|}{k + |\lambda|k^2}$, where λ is defined as in Theorem 1, it is sufficient to show that

$$\frac{k(1 - |\lambda|)(|\lambda| + k^2) + k^2(n - 1)|\mu - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda|k^2) + k(n - 1)|\mu - \lambda^2|} \geq \frac{k + |\lambda|}{k + |\lambda|k^2},$$

which implies

$$(k-1) [(1-|\lambda|)\{k^3|\lambda|(k+1) + k^2|\lambda|^2 + k^3 + |\lambda|(k+1)\} \\ + k(n-1)|\mu - \lambda^2|\{k + |\lambda|(k^2 + k + 1)\}] \geq 0,$$

from which we eventually obtain

$$(1-|\lambda|) \{k^4|\lambda| + k^3(|\lambda| + 1) + k^2|\lambda|^2 + k|\lambda| + |\lambda|\} \\ + k(n-1)|\mu - \lambda^2| \{k^2|\lambda| + k(|\lambda| + 1) + |\lambda|\} \geq 0,$$

which obviously holds due to the fact that $|\lambda| \leq 1$ (by Lemma 2.1).

Remark 3. If we let $\gamma \rightarrow \infty$ in inequality (1.20) of Theorem 1, we obtain

$$\|p'\|_\infty \leq \frac{n}{C+1} \|p\|_\infty.$$

Using (1.21), the above inequality becomes inequality (1.17) of Theorem A.

Remark 4. Inequality (1.20) in its ordinary form obtained as in Remark 3 improves inequality (1.11) due to Govil et al. [8] as shown in Remark 2. Hence, it also improves (1.9) due to Malik [11]. For the sake of completeness, we show that

$$\frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \leq \frac{n}{1+k},$$

i.e.

$$\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \leq 1,$$

for which it is sufficient to show that

$$1 + k^2|\lambda| \leq 1 - k + k^2 + k|\lambda|, \quad \text{where } k \geq 1 \text{ and } |\lambda| \leq 1,$$

which implies that

$$k(1-|\lambda|) \leq k^2(1-|\lambda|), \\ \text{i.e. } k \leq k^2, \\ \text{i.e. } 1 \leq k, \quad \text{which is true.}$$

Next, we obtain the following L^γ analogue of Theorem B.

Theorem 2. If $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $\gamma > 0$

$$n \left\| \frac{p}{p'} \right\|_\gamma \leq \|1 + Dz\|_\gamma \quad (1.22)$$

where

$$D = k \frac{(1 - |\omega|)(|\omega| + k^2) + k(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 + k^2|\omega|) + k(n-1)|\Omega - \omega^2|}, \quad (1.23)$$

$$\omega = \frac{1}{nk} \frac{\bar{a}_{n-1}}{\bar{a}_n} \quad \text{and} \quad \Omega = \frac{2}{n(n-1)k^2} \frac{\bar{a}_{n-2}}{\bar{a}_n}.$$

Remark 5. We claim that $D \leq t_{k,1}$, where D and $t_{k,1}$ are defined by 1.23 in Theorem 2 and inequality (1.15) respectively. Since $t_{k,1} = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|} = \frac{k + |\omega|}{k + k^2|\omega|}$, $D \leq t_{k,1}$ implies

$$\frac{k(1 - |\omega|)(|\omega| + k^2) + k^2(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 + |\omega|k^2) + k(n-1)|\Omega - \omega^2|} \leq \frac{k + |\omega|}{k + |\omega|k^2}.$$

The above inequality simplifies to

$$(k-1) \left[(1 - |\omega|) \{ k^3|\omega|(k+1) + k^2|\omega|^2 + k^3 + |\omega|(k+1) \} \right. \\ \left. + k(n-1)|\mu - \omega^2| \{ k + |\omega|(k^2 + k + 1) \} \right] \leq 0.$$

Since $k \leq 1$, we have from the above inequality

$$(1 - |\omega|) \{ k^3|\omega|(k+1) + k^2|\omega|^2 + k^3 + |\omega|(k+1) \} \\ + k(n-1)|\mu - \omega^2| \{ k + |\omega|(k^2 + k + 1) \} \geq 0,$$

which clearly holds as $|\omega| \leq 1$ due to Lemma 2.2. Thus, inequality (1.22) is an improvement of (1.15) due to Aziz and Rather [2].

Since $|p'(e^{i\theta})| \leq \|p'\|_\infty$ for $0 \leq \theta < 2\pi$, the following corollary immediately follows.

Corollary 3. If $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $\gamma > 0$

$$\|p'\|_\infty \geq \frac{n}{\|1 + Dz\|_\gamma} \|p\|_\gamma, \quad (1.24)$$

where D is defined by (1.23) in Theorem 2.

Remark 6. Due to the fact that $D \leq t_{k,1}$, inequality (1.24) of Corollary 3 is a refinement of inequality (1.16) due to Aziz and Rather [2].

Remark 7. Corollary 3 is a generalization of Theorem B. We know by definition

$$\frac{n}{\|1 + Dz\|_\infty} = n \lim_{\gamma \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + De^{i\theta}|^\gamma d\theta \right\}^{-\frac{1}{\gamma}} = \frac{n}{1 + D},$$

its R.H.S. on using (1.23) further simplifies to

$$\frac{n}{1 + k} \frac{(1 - |\omega|)(1 + k^2|\omega|) + k(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 - k + k^2 + k|\omega|) + k(n-1)|\Omega - \omega^2|}. \quad (1.25)$$

Thus, letting $\gamma \rightarrow \infty$, (1.24) reduces to (1.18) of Theorem B.

Remark 8. The ordinary inequality form 1.18 of (1.24) obtained by taking the limit as $\gamma \rightarrow \infty$ in (1.24) is an improvement of inequality (1.13) due to Malik [11]. To verify this, it is sufficient to show that, for $|\omega| \leq 1$ and $k \leq 1$

$$\frac{(1 - |\omega|)(1 + k^2|\omega|) + k(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 - k + k^2 + k|\omega|) + k(n-1)|\Omega - \omega^2|} \geq 1,$$

which is equivalent to showing

$$\begin{aligned} 1 + k^2|\omega| &\geq 1 - k + k^2 + k|\omega|, \\ \text{i.e. } 1 &\geq k, \quad \text{which is true as } |\omega| \leq 1. \end{aligned}$$

Finally, we use Holder's inequality to establish the following result which is a generalization of Theorem 2.

Theorem 4. If $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $\gamma > 0$, $r > 1$, $s > 1$ with $r^{-1} + s^{-1} = 1$

$$\|p'\|_{s\gamma} \geq \frac{n}{\|1 + Dz\|_{r\gamma}} \|p\|_\gamma, \quad (1.26)$$

where D is given by (1.23) in Theorem 2.

Letting $s \rightarrow \infty$ (so that $r \rightarrow 1$) in Theorem 4, we obtain Corollary 3, and hence consequences of Corollary 3 follows. Thus, taking the limit as $s \rightarrow \infty$ (so that $r \rightarrow 1$) or $r \rightarrow \infty$ (so that $s \rightarrow 1$) and then letting $\gamma \rightarrow \infty$, in view of Remark 6, (1.26) of Theorem 4 reduces to (1.18) of Theorem B.

2. Lemmas

We require the following lemmas to prove our results.

The following lemma is due to Govil et al. [8].

Lemma 2.1. *If $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k \geq 1$, then*

$$C|p'(z)| \leq |q'(z)|, \quad (2.1)$$

where here and elsewhere $q(z) = z^n p\left(\frac{1}{\bar{z}}\right)$,

$$C = \frac{k(1 - |\lambda|)(|\lambda| + k^2) + k(n-1)|\mu - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda|k^2) + k(n-1)|\mu - \lambda^2|}, \quad (2.2)$$

$$\lambda = \frac{k}{n} \frac{a_1}{a_0} \quad \text{and} \quad \mu = \frac{2k^2}{n(n-1)} \frac{a_2}{a_0}$$

such that

$$|\lambda| \leq 1 \quad \text{and} \quad (n-1)|\mu - \lambda^2| \leq 1 - |\lambda|^2.$$

Lemma 2.2. *If $p \in \mathbb{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then on $|z| = 1$*

$$|q'(z)| \leq D|p'(z)|, \quad (2.3)$$

where

$$D = k \frac{(1 - |\omega|)(|\omega| + k^2) + k(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 + k^2|\omega|) + k(n-1)|\Omega - \omega^2|},$$

$$\omega = \frac{1}{nk} \frac{\bar{a}_{n-1}}{\bar{a}_n} \quad \text{and} \quad \Omega = \frac{2}{n(n-1)k^2} \frac{\bar{a}_{n-2}}{\bar{a}_n} \quad \text{such that } |\omega| \leq 1 \quad \text{and} \quad (n-1)|\Omega - \omega^2| \leq 1 - |\omega|^2.$$

Proof of Lemma 2.2. Since $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then the polynomial $q(z) = z^n p\left(\frac{1}{\bar{z}}\right)$ has no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Thus, applying Lemma 2.1 to the polynomial $q(z)$, we have

$$|q'(z)| \leq k \frac{(1 - |\omega|) \left(1 + |\omega| \frac{1}{k^2}\right) + \frac{1}{k}(n-1)|\Omega - \omega^2|}{(1 - |\omega|) \left(|\omega| + \frac{1}{k^2}\right) + \frac{1}{k}(n-1)|\Omega - \omega^2|} |p'(z)|,$$

where

$$\omega = \frac{1/k}{n} \frac{\bar{a}_{n-1}}{\bar{a}_n} = \frac{1}{nk} \frac{\bar{a}_{n-1}}{\bar{a}_n}$$

and

$$\Omega = \frac{2/k^2}{n(n-1)} \frac{\bar{a}_{n-2}}{\bar{a}_n} = \frac{2}{n(n-1)k^2} \frac{\bar{a}_{n-2}}{\bar{a}_n}.$$

Then,

$$|q'(z)| \leq k \frac{(1 - |\omega|)(|\omega| + k^2) + k(n-1)|\Omega - \omega^2|}{(1 - |\omega|)(1 + k^2|\omega|) + k(n-1)|\Omega - \omega^2|} |p'(z)|,$$

and $|\omega| \leq 1$ and $(n-1)|\Omega - \omega^2| \leq 1 - |\omega|^2$. □

Lemma 2.3. If $p \in \mathbb{P}_n$ and $q(z) = z^n p\left(\frac{1}{\bar{z}}\right)$, then for each α , $0 \leq \alpha < 2\pi$ and $\gamma > 0$

$$\int_0^{2\pi} \int_0^{2\pi} \left| q'(e^{i\theta}) + e^{i\alpha} p'(e^{i\theta}) \right|^\gamma d\theta d\alpha \leq 2\pi n^\gamma \int_0^{2\pi} \left| p(e^{i\theta}) \right|^\gamma d\theta. \quad (2.4)$$

The above lemma is due to Aziz and Rather [2].

Lemma 2.4. Let z be any complex and independent of α , where α is any real, then for $\gamma > 0$

$$\int_0^{2\pi} |1 + ze^{i\alpha}|^\gamma d\alpha = \int_0^{2\pi} |e^{i\alpha} + z|^\gamma d\alpha \quad (2.5)$$

The above lemma is due to Gardner and Govil [5].

3. Proof of the Theorems

Proof of Theorem 1. Since $p(z)$ has no zero in $|z| < k$, $k \geq 1$, we have, by Lemma 2.1

$$C|p'(z)| \leq |q'(z)| \quad \text{for } |z| = 1, \quad (3.1)$$

where C is defined by (1.21) in Theorem 1.

We have for every real α and $R \geq r \geq 1$,

$$|R + e^{i\alpha}| \geq |r + e^{i\alpha}|.$$

Then, for every $\gamma > 0$ we have

$$\int_0^{2\pi} |R + e^{i\alpha}|^\gamma d\alpha \geq \int_0^{2\pi} |r + e^{i\alpha}|^\gamma d\alpha. \quad (3.2)$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $p'(e^{i\theta}) \neq 0$, we denote $R = \left| \frac{q'(e^{i\theta})}{p'(e^{i\theta})} \right|$ and $r = C$, then by (3.1) and Remark 1 we have $R \geq r \geq 1$ and for each $\gamma > 0$,

$$\begin{aligned} \int_0^{2\pi} \left| q'(e^{i\theta}) + e^{i\alpha} p'(e^{i\theta}) \right|^\gamma d\alpha &= \left| p'(e^{i\theta}) \right|^\gamma \int_0^{2\pi} \left| \frac{q'(e^{i\theta})}{p'(e^{i\theta})} + e^{i\alpha} \right|^\gamma d\alpha \\ &= \left| p'(e^{i\theta}) \right|^\gamma \int_0^{2\pi} \left| \left| \frac{q'(e^{i\theta})}{p'(e^{i\theta})} \right| + e^{i\alpha} \right|^\gamma d\alpha \\ &\quad \text{(by Lemma 2.4)} \\ &= \left| p'(e^{i\theta}) \right|^\gamma \int_0^{2\pi} |R + e^{i\alpha}|^\gamma d\alpha \\ &\geq \left| p'(e^{i\theta}) \right|^\gamma \int_0^{2\pi} |r + e^{i\alpha}|^\gamma d\alpha \\ &\quad \text{(by (3.2)).} \end{aligned}$$

Hence,

$$\int_0^{2\pi} \left| q'(e^{i\theta}) + e^{i\alpha} p'(e^{i\theta}) \right|^\gamma d\alpha \geq \left| p'(e^{i\theta}) \right|^\gamma \int_0^{2\pi} |C + e^{i\alpha}|^\gamma d\alpha \quad (\because r = C), \quad (3.3)$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $p'(e^{i\theta}) \neq 0$.

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $p'(e^{i\theta}) = 0$, inequality (3.3) trivially holds. Hence, using (3.3) in Lemma 2.3, we obtain for each $\gamma > 0$,

$$\int_0^{2\pi} |p'(e^{i\theta})|^\gamma d\theta \int_0^{2\pi} |C + e^{i\alpha}|^\gamma d\alpha \leq 2\pi n^\gamma \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta,$$

which is equivalent to

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n \left\{ \frac{1}{2\pi} \int_0^{2\pi} |C + e^{i\alpha}|^\gamma d\alpha \right\}^{-\frac{1}{\gamma}} \left\{ \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \quad (3.4)$$

from which the desired conclusion of Theorem 1 follows. $\square$

Proof of Theorem 2. Since $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, $p'(z)$ also has all its zeros in $|z| \leq k$, $k \leq 1$. Hence, by Gauss-Lucas Theorem, the polynomial

$$z^{n-1} \overline{p\left(\frac{1}{\bar{z}}\right)} = nq(z) - zq'(z) \quad (3.5)$$

has all its zeros in $|z| \geq \frac{1}{k}$, $\frac{1}{k} \geq 1$.

Further, since $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, we have by Lemma 2.2

$$|q'(z)| \leq D|p'(z)| \quad \text{on } |z| = 1, \quad (3.6)$$

where D is given by (1.23).

For $|z| = 1$, we also have

$$|p'(z)| = |nq(z) - zq'(z)|. \quad (3.7)$$

Using (3.7) in 3.6 we have on $|z| = 1$

$$|q'(z)| \leq D|nq(z) - zq'(z)|. \quad (3.8)$$

Thus, in view of (3.5) and (3.8), the function

$$\phi(z) = \frac{zq'(z)}{D\{nq(z) - zq'(z)\}}$$

is analytic in $|z| \leq 1$, $|\phi(z)| \leq 1$ on $|z| = 1$ and $\phi(0) = 0$. Therefore, the function $1 + D\phi(z)$ is subordinate to the function $1 + Dz$ for $|z| \leq 1$. Hence, by a well known property of subordination [9], we have for each $\gamma > 0$

$$\int_0^{2\pi} \left| 1 + D\phi(e^{i\theta}) \right|^\gamma d\theta \leq \int_0^{2\pi} \left| 1 + De^{i\theta} \right|^\gamma d\theta. \quad (3.9)$$

Now,

$$1 + D\phi(z) = 1 + \frac{zq'(z)}{nq(z) - zq'(z)} = \frac{nq(z)}{nq(z) - zq'(z)},$$

which implies for $|z| = 1$,

$$\begin{aligned} |nq(z)| &= |1 + D\phi(z)| |nq(z) - zq'(z)| \\ &= |1 + D\phi(z)| |p'(z)| \quad (\text{by (3.7)}). \end{aligned}$$

Since $|p(z)| = |q(z)|$ on $|z| = 1$, we have from the preceding inequality

$$n|p(z)| = |1 + D\phi(z)| |p'(z)| \quad \text{on } |z| = 1 \quad (3.10)$$

i.e.,

$$n \left| \frac{p(z)}{p'(z)} \right| = |1 + D\phi(z)| \quad \text{on } |z| = 1.$$

Then for each $\gamma > 0$ and $0 \leq \theta < 2\pi$, we have

$$n^\gamma \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{p'(e^{i\theta})} \right|^\gamma d\theta = \int_0^{2\pi} \left| 1 + D\phi(e^{i\theta}) \right|^\gamma d\theta,$$

which on using (3.9) gives

$$n^\gamma \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{p'(e^{i\theta})} \right|^\gamma d\theta \leq \int_0^{2\pi} \left| 1 + De^{i\theta} \right|^\gamma d\theta,$$

which implies

$$n \left\{ \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{p'(e^{i\theta})} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq \left\{ \int_0^{2\pi} \left| 1 + De^{i\theta} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}},$$

which equivalently gives inequality (1.22). □

Proof of Theorem 4. Proceeding similarly as in the proof of Theorem 2, we have from (3.10)

$$n|p(z)| = |1 + D\phi(z)||p'(z)|.$$

Then, for each $\gamma > 0$ and $0 \leq \theta < 2\pi$ we have

$$n^\gamma \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta = \int_0^{2\pi} |1 + D\phi(e^{i\theta})|^\gamma |p'(e^{i\theta})|^\gamma d\theta.$$

Applying Holder's inequality to the above inequality, we have for $r > 1$, $s > 1$ with $r^{-1} + s^{-1} = 1$ and for each $\gamma > 0$

$$n^\gamma \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \leq \left\{ \int_0^{2\pi} |1 + D\phi(e^{i\theta})|^{r\gamma} d\theta \right\}^{\frac{1}{r}} \left\{ \int_0^{2\pi} |p'(e^{i\theta})|^{s\gamma} d\theta \right\}^{\frac{1}{s}},$$

which implies

$$n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq \left\{ \int_0^{2\pi} |1 + D\phi(e^{i\theta})|^{r\gamma} d\theta \right\}^{\frac{1}{r\gamma}} \left\{ \int_0^{2\pi} |p'(e^{i\theta})|^{s\gamma} d\theta \right\}^{\frac{1}{s\gamma}}.$$

Using (3.9) in the above inequality, we have

$$n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq \left\{ \int_0^{2\pi} |1 + De^{i\theta}|^{r\gamma} d\theta \right\}^{\frac{1}{r\gamma}} \left\{ \int_0^{2\pi} |p'(e^{i\theta})|^{s\gamma} d\theta \right\}^{\frac{1}{s\gamma}},$$

which equivalently gives

$$\|p'\|_{s\gamma} \geq \frac{n}{\|1 + Dz\|_{r\gamma}} \|p\|_\gamma,$$

which is the desired conclusion of the theorem. $\square$

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