A GENERALIZATION OF TRIPLE STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

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Abstract: In this paper, we introduce a class of summability methods that can be applied to λ-triple statistical convergence in topological groups and we show some interesting results.

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1. Introduction and definitions preliminaries

Looking through historically at statistical convergence of single sequences, we shall recall that the notion of statistical convergence of sequences was first studied by Fast [3]. The notion of statistical convergence of a sequence \((x_n)\) in a locally convex Hausdorff topological linear space \(X\) was presented recently by Maddox [8], where it was shown that the slow oscillation of \((s_n)\) was a Tauberian condition for the statistical convergence of \((s_n)\). In [7], statistical convergence to normed spaces was extended by Kolk. Further in [1] and [2], Cakalli extended

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this notation to topological Hausdorff groups. The study of triple sequence in different fields of sequences spaces has grown in the last decade (see [4, 5, 6]).

By the convergence of a triple sequence, we mean the convergence in Pringsheim’s sense [9]. A triple sequence \( x = (x_{asd}) \) is said to be convergent in the Pringsheim’s sense if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{asd} - \psi| < \varepsilon \) whenever \( a, s, d \geq N \). \( \psi \) is called the Pringsheim limit of \( x \). Furthermore, A triple sequence \( (x = x_{asd}) \) is said to be Cauchy sequence if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( x_{asd} - \psi \in V \) whenever \( a, s, d \geq N \). \( \psi \) is called the Pringsheim limit of \( x \). A triple sequence \( x = (x_{asd}) \) is said to be a Cauchy sequence if for every neighbourhood \( V \) of 0 there exists \( N \in \mathbb{N} \) such that \( x_{pql} - x_{asd} \in V \) for all \( p \geq a \geq N, q \geq s \geq N \) and \( l \geq d \geq N \). In a topological group \( E \), the above definitions become as in the following: a triple sequence \( x = (x_{asd}) \) in \( E \) is said to be convergent to \( \psi \) in \( E \) in the Pringsheim’s sense if for every neighbourhood \( V \) of 0 there exists \( N \in \mathbb{N} \) such that \( x_{asd} - \psi \in V \) whenever \( a, s, d \geq N \). \( \psi \) is called the Pringsheim limit of \( x \).

By \( E \), we will denote an Abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset \( B \) of \( E \), \( s(B) \) will denote the set of all sequences \( (x_a) \) such that \( (x_a) \) is in \( B \) for \( a = 1, 2, 3, ... \) \( c(E) \) will denote the set of all convergent sequences. On the other hand, a sequence \( (x_a) \) in \( E \) is called statistically convergent to an element \( \psi \) of \( E \) if for each neighbourhood \( V \) of 0, (see [2]) \( \lim_{a \to \infty} \frac{1}{a} |z \leq a : x_a - \psi \notin V| = 0 \), and is called statistically Cauchy in \( E \) if for each neighbourhood \( V \) of 0 there exists a positive integer \( a_0(V) \), depending on the neighbourhood \( V \), such that \( \lim_{a \to \infty} \frac{1}{a} |z \leq a : x_a - x_{a_0(V)} \notin V| = 0 \) where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in \( E \) is denoted by \( S(E) \) and the set of all statistically Cauchy sequences in \( E \) is denoted by \( SC(E) \). It is known that \( SC(E) = S(E) \) if \( E \) is complete. Additionally, those notions and the notion \( \lambda \)-statistical convergent were extended for double sequences by Savas [11].

On the other hand, let \( \lambda = (\lambda_p, \mu = (\lambda_q)) \) and \( \phi = (\phi_l) \) be three non-decreasing sequences of positive real numbers, three of them of which tends to \( \infty \) as \( p, q \) and \( l \) approach \( \infty \), respectively. Besides, let \( \lambda_{p+1} \leq \lambda_p + 1, \lambda_1 = 1, \mu_{q+1} \leq \mu_q + 1, \mu_1 = 1 \) and \( \phi_{l+1} \leq \phi_l + 1, \phi_1 = 1 \). The collection of such sequence will be denoted by \( \Delta \). We write the generalized double de la Valée-Poussin mean by

\[
t_{p,q,l}(x) = \frac{1}{\lambda_p \mu_q \phi_l} \sum_{a \in I_p, s \in J_q, d \in W_l} x_{asd},
\]

where \( I_p = [p - \lambda_p + 1, p], J_Q = [Q - mu_q + 1, q] \) and \( W_l = [l - \phi_l + 1, l] \).
Throughout this paper we shall denote \( \lambda_p \mu_q \phi_l \) by \( \lambda_{pql} \) and \((a \in I_p, s \in J_q, d \in W_l) \) by \((a, s, d) \in I_{pql} \).

The aim of this paper is to introduce the \( \lambda \)-triple statistical convergence of triple sequences in topological groups and to prove some useful theorems.

2. \( \lambda \)-Triple statistical convergence

Let \( J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) be a three-dimensional set of positive integers and let \( J(q, w, e) \) be the numbers of \((a, s, d)\) in \( J \) such that \( a \leq q, s \leq w \) and \( d \leq e \). Then, the three-dimensional analogue of natural density can be defined as follows. The lower asymptotic density of a set \( I \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) is defined as

\[
\delta_3(I) = \lim_{q,w,e} \inf J(q, w, e)
\]

In case that the sequence \((J(q, w, e)/qwe)\) has a limit in Pringsheim’s sense, then we say that \( J \) has a triple natural density and is defined as

\[
\delta_3(J) = \lim_{q,w,e} J(q, w, e)/qwe.
\]

Sahiner and Tripathy [10] called a real triple sequence \( x = (x_{asd}) \) statistically convergent to the number \( \psi \) if for each \( \varepsilon > 0 \), the set \( \{(a, s, d), a \leq q, s \leq w \text{ and } d \leq e : |x_{asd} - \psi| \geq \varepsilon\} \) has triple natural density zero. In this case, we write \( S_3\)-lim_{a,s,d} x_{asd} = \psi \) and we denote the set of all statistically convergent triple sequences by \( S_3 \). Now, we define statistical convergence of triple sequences \( x = (x_{asd}) \) in a topological group in the following.

**Definition 1.** A triple sequence \( x = (x_{asd}) \) is statistically convergent to a point \( \psi \) of \( E \) if for each neighbourhood \( V \) of 0 the set

\[
\{(a, s, d), a \leq q, s \leq w \text{ and } d \leq e : x_{asd} - \psi / \notin V\}
\]

has a triple natural density zero. In this case, we write \( S_3(E)\)-lim_{a,s,d} x_{asd} = \psi \) and we write the set of all statistically convergent triple sequences by \( S_3(E) \).

**Definition 2.** A triple sequence \( x = (x_{asd}) \) is said to be \( S^{3}_{\lambda} \)-convergent to \( \psi \) of \( E \) (or \( \lambda \)-triple statistically convergent to \( \psi \) of \( E \)) of for each neighbourhood \( V \) of 0, the set

\[
\{(a, s, d) \in I_{pql} : x_{asd} - \psi / \notin V\}
\]
has triple natural density zero. In this case we write \( S_\lambda^3 \text{-} \lim_{a,s,d \to \infty} x_{asd} = \psi \)

or \( x_{asd} \to \psi(S_\lambda^3) \), and we write the set of all \( \lambda \)-statistically convergent triple sequences by \( S_\lambda^3(E) \).

**Remark 3.** A \( \lambda \)-statistically convergent triple sequence has a unique limit, i.e. if \( x \) is \( \lambda \)-statistically convergent to elements \( \psi_1 \) and \( \psi_2 \) of \( E \), then \( \psi_1 = \psi_2 \).

**Theorem 4.** A triple sequence \( x = (x_{asd}) \) in \( E \) is \( \lambda \)-triple statistically convergent to \( \psi \) if and only if there exists a subset \( J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) such that \( \delta_\lambda^3(J) = 1 \) and \( \lim_{a,s,d \to \infty} x_{asd} = \psi \) where limit is being taken over the set \( E \), i.e. \( (a,s,d) \in E \).

**Proof.** Necessity: Let us suppose that \( x \) be \( \lambda \)-triple statistically convergent to \( \psi \), and \( (V_r) \) be a base of nested closed neighbourhood of 0. Now, write \( J_r = \{(a,s,d) \in I_{pql} : x_{asd} - \psi \notin V_r \} \) and \( Q_r = \{(a,s,d) \in I_{pql} : x_{asd} - \psi \in V_r \} \) where \( r = 1,2,3,\ldots \). Then, \( \delta_\lambda^3(J_r) = 0 \) and

\[
Q_1 \supset Q_2 \supset \ldots \supset Q_a \supset Q_{a+1} \supset \ldots \tag{1}
\]

and

\[
\delta_\lambda^3(Q_r) = 1, r = 1,2,3,\ldots \tag{2}
\]

Now, we have to show that for \( (a,s,d) \in Q_r, (x_{asd}) \) is \( \lambda \)-triple statistically convergent to \( \psi \). Now, consider that \( (x_{asd}) \) is not \( \lambda \)-triple statistically to \( \psi \) so that there is a neighbourhood \( V \) of 0 such that \( x_{asd} - \psi \notin V \) for infinitely many terms. Let \( V_r \subset V \) where \( r = 1,2,3,\ldots \) and \( Q_V = \{(a,s,d) : x_{asd} - \psi \notin V \} \). Then, \( \delta_\lambda^3(Q_V) = 0 \) and by (1), \( Q_r \subset Q_V \). Therefore, \( \delta_\lambda^3(Q_r) = 0 \) which is a contradiction to (2). Hence, \( (x_{asd}) \) is \( \lambda \)-triple statistically convergent to \( \psi \).

Sufficiency: Consider that there exists a subset \( J = \{(a,s,d) \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \} \) such that \( \delta_\lambda^3(J) = 1 \) and \( \lim_{a,s,d} x_{asd} = \psi \), i.e. there exists an \( r_0 \in \mathbb{N} \) such that for each neighbourhood \( V \) of 0, \( x_{asd} - \psi \notin V \) for every \( a,s,d \geq r_0 \). Now,

\[
J_V = \{(a,s,d) : x_{asd} - \psi \notin V \}
\]

\[
\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(a_{r_0+1}, s_{r_0+1}, d_{r_0+1}), (a_{r_0+2}, s_{r_0+2}, d_{r_0+2}), \ldots \}.
\]

Therefore, \( \delta_\lambda^3(J_V) \leq 0 \). It follows that \( x \) is \( \lambda \)-triple statistically convergent to \( \psi \). \( \square \)

**Corollary 5.** If a triple sequence \( (x_{asd}) \) is \( \lambda \)-triple statistically convergent to \( \psi \). Then, there exists a triple sequence \( (y_{asd}) \) such that \( \lim_{a,s,d} y_{asd} = \psi \) and \( \delta_\lambda^3 \{(a,s,d) : x_{asd} = y_{asd} \} = 1 \), i.e. \( x_{asd} = y_{asd} \) for almost all \( a,s,d \).
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Definition 6. In a topological group, triple sequence \( x = (x_{asd}) \) is called \( \lambda \)-triple statistically Cauchy if for each neighbourhood \( V \) of 0 there exists \( G = G(V), H = H(V) \) and \( Q = Q(V) \) such that for all \( a, q \geq G, s, w \geq H \) and \( d, e \geq Q \) the set \( \{(a, s, d) \in I_{pql} : x_{asd} - x_{qwe} \notin V\} \) has triple natural density zero. In this case, we denote the set of all statistically Cauchy triple sequences by \( S^3C(E) \).

Theorem 7. Let \( E \) be complete. A triple sequence \( x = (x_{asd}) \) in \( E \) is \( \lambda \)-triple statistically convergent if and only if \( x \) is \( \lambda \)-triple statistically Cauchy.

Proof. Let \( x = (x_{asd}) \) be \( \lambda \)-triple statistically convergent to \( \psi \). Let \( V \) be any neighbourhood of 0. Then, we can choose a symmetric neighbourhood \( W \) of 0 such that \( W + W \subset V \). Then for this neighbourhood \( W \) of 0, the set \( \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin W\} \) has triple natural \( \lambda \)-density 0. For each neighbourhood \( V \) of 0, the set \( \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin V\} \) has triple natural \( \lambda \)-density zero- Then, we can choose numbers \( G, H \) and \( Q \) such that \( x_{GHQ} - \psi \notin V \). Now, we write \( T_V = \{(a, s, d) \in I_{pql} : x_{asd} - x_{GHQ} \notin V\}, L_W = \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin W\} \) and \( K_W = \{(N, M, J) \in I_{pql} : x_{GHQ} - \psi \notin W\} \). Then, \( T_V \subset L_W \cup K_W \) and hence \( \delta^3(\lambda)(T_V) \leq \delta^3(\lambda)(L_W) + \delta^3(\lambda)(K_W) = 0 \). Therefore, we obtain that \( x \) is \( \lambda \)-triple statistically Cauchy. To prove the converse suppose that there is a \( \lambda \)-triple statistically Cauchy sequence \( x \) but it is not \( \lambda \)-triple statistically convergent. Then we can find natural numbers \( G, H \) and \( Q \) such that the set \( T_V \) has triple natural \( \lambda \)-density zero. It follows from this that the set \( Z_V = \{(a, s, d) \in I_{pql} : x_{asd} - X_{GHQ} \in V\} \) has triple natural density 1. Now, we can choose a neighbourhood \( W \) of 0 such that \( W + W \subset V \). Now, take any fixed non-zero element \( \psi \) of \( E \). Let \( x_{asd} - X_{GHQ} = x_{asd} - \psi + \psi - X_{GHQ} \). It follows from this equality that \( x_{asd} - x_{GHQ} \in V \) if \( x_{asd} - \psi \in W \). Since \( x \) is not \( \lambda \)-triple statistically convergent to \( \psi \), the set \( L_W \) has triple natural density 1, i.e. the set \( \{(a, s, d)a \leq q, s \leq w, d \geq e : x_{asd} - \psi \notin W\} \) has triple natural density 0. Hence the set \( \{(a, s, d)a \leq q, s \leq w, d \leq e : x_{asd} - x_{GHQ} \in W\} \) has triple natural density 0, i.e. the set \( T_V \) has triple natural density 1 which is a contradiction. 

Taking into account Theorems 4 and 7, we can state the following theorem and the proof is following directly by the previous results.

Theorem 8. If \( E \) is complete, then the following conditions are equivalent:

1. \( x \) is \( \lambda \)-triple statistically convergent to \( \psi \),
2. $x$ is $\lambda$-triple statistically Cauchy,

3. there exists a subsequence $y$ of $x$ such that $\lim_{a,s,d} y_{asd} = \psi$.

References


