NEW SUBCLASSES OF BAZILEVIĆ CLOSE-TO-CONVEX MAPS

K.O. Babalola¹, O.Y. Saka-Balogun²§

¹ Department of Mathematics
University of Ilorin
Ilorin, NIGERIA

² Department of Mathematical and Physical Sciences
Afe Babalola University
Ekiti, NIGERIA

Abstract: We introduce and study new interesting subclasses of the class of Bazilević close-to-convex maps. The functions in the new class are said to be $\lambda$-pseudo-close-to-convex. Properties such as univalence, integral representations, inclusion conditions and coefficient inequalities were investigated in these new subclasses. We provide also some examples.

AMS Subject Classification: 30C45, 30C50

Key Words: $\lambda$-pseudo-close-to-convex; $\lambda$-pseudo-starlike; Bazilević close-to-convex

1. Introduction

Let $A$ be the class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots$$

which are regular in the unit disk $E = \{z : |z| < 1\}$, and we denote by $P_{\beta}$ the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

Received: October 12, 2021

© 2022 Academic Publications

§Correspondence author
which are also regular in the unit disk $E$ and satisfy $\text{Re } p(z) > \beta, \ 0 \leq \beta < 1$.

For $0 \leq \beta < 1$, a function $f \in A$ is said to be starlike and close-to-convex of order $\beta$ respectively if and only if $\text{Re } zf'(z)/f(z) > \beta$ and $\text{Re } zf''(z)/g(z) > \beta$ (for $g(z)$ starlike in $E$). Using the usual notations, we denote the two classes of functions respectively by $S^*(\beta)$ and $K(\beta)$. These classes of functions both consist only of univalent functions in $E$. In [6], Singh studied a class univalent maps, here in this paper referred to as the Bazilević close-to-convex of type $\alpha$, consisting functions that satisfy the geometric condition:

$$\text{Re } \frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}} > \beta, \ z \in E,$$

where $\alpha$ is a nonnegative real number. This class is denoted by $B(\alpha)$, and it is a special case of the well known generally univalent Bazilević maps (see [6]). This class $B(\alpha)$ includes the starlike and close-to-convex functions as the cases $\alpha = 0$ and $\alpha = 1$ respectively. For $0 \leq \beta < 1$, we extend $B(\alpha)$ to Bazilević close-to-convex of type $\alpha$, order $\beta$, denoted by $B(\alpha, \beta)$ and consisting of functions satisfying

$$\text{Re } \frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}} > \beta, \ z \in E.$$

In the sequel, all the powers are meant as principal values.

**Definition 1.** Let $f \in A$, and $g \in S^*(\equiv S^*(0))$. Suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ are real. Then $f(z)$ belongs to the class of $\lambda$-pseudo-close-to-convex functions, denoted by $SB_\lambda(\beta)$, if and only if

$$\text{Re } \frac{(zf'(z))^\lambda}{g(z)^{\lambda-1}f(z)} > \beta, \ z \in E.$$

(a) If $\lambda = 1$, then $SB_1(\beta)$ is the class $S^*(\beta)$ of starlike functions of order $\beta$.

(b) For $\lambda = 2$, we note that functions in $SB_2(\beta)$ are defined by

$$\text{Re } \frac{zf'(z)zf'(z)}{f(z)g(z)} > \beta, \ z \in E,$$

which is a product combination of geometric expressions for starlike and close-to-convex functions.

(c) In general, the geometric definition of $SB_\lambda(\beta)$ can be put in the form

$$\text{Re } \frac{zf'(z)}{f(z)} \left( \frac{zf'(z)}{g(z)} \right)^\gamma > \beta, \ \gamma = \lambda - 1 > 0$$
which informs for the nomenclature of $\lambda$-pseudo-close-to-convexity for the new class of functions.

(d) If the starlike function $g(z)$ satisfies the identity map $g(z) = z$, then $SB_\lambda(\beta)$ becomes the class of $\lambda$-pseudo-starlike functions of order $\beta$, introduced and studied in [2]. The examples given in [2] are a quick indication that the new concept is nonvoid. For nontrivial $g(z)$, we give more examples later.

We prove in Section 3 that $SB_\lambda(\beta)$ is a subclass of Bazilevič close-to-convex maps of type $\alpha$, order $\beta$. We also obtain some characterizations of the new class of functions. In particular, we obtain their integral representations, sufficient inclusion condition, some coefficient bounds and also estimates of their Feketo-Szegö functional.

In the next section, we state the preliminary results, which we shall employ in our proofs in Section 3.

2. Preliminary lemmas

The following lemmas will be useful in proving our main results.

**Lemma 1.** ([3]) Let $q(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P(= P_0)$. Then $|p_k| \leq 2, k = 1, 2, 3, \ldots$. Equality is attained by the Möbius function $L_0(z) = (1 + z)/(1 - z)$.

**Lemma 2.** ([1]) Let $q(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$. Then we have the sharp inequalities

$$
|p_2 - \sigma p_1^2/2| \leq 2 \max\{1, |\sigma - 1|\}.
$$

Now if $p = 1 + c_1 z + c_2 z^2 + \cdots \in P_\beta$, then $p(z) = \beta + (1 - \beta)q(z) = 1 + (1 - \beta)p_1 z + (1 - \beta)p_2 z^2 + \cdots$, so that Lemmas 1 and 2 re-present as:

**Lemma 3.** Let $p \in P_\beta$. Then $|c_k| \leq 2(1 - \beta), k = 1, 2, 3, \ldots$. Equality is attained by $L_{0, \beta}(z) = [1 + (1 - 2\beta)z]/(1 - z)$.

**Lemma 4.** Let $p \in P_\beta$. Then we have the sharp inequalities

$$
|c_2 - \sigma c_1^2/2| \leq 2(1 - \beta) \max\{1, |(1 - \beta)\sigma - 1|\}.
$$
Also we shall need the following lemmas.

**Lemma 5.** ([2]) Let \( p \in P_\beta \). If \( q(z) = [p(z)]^t \) for \( t \in [0, 1] \), then \( q(0) = 1 \) and \( \Re q(z) > \beta^t \).

**Lemma 6.** ([2, 4]) Let \( p \in P \). Suppose that

\[
\Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\beta - 1}{2\beta}, \quad z \in E,
\]

then \( \Re p(z) > 2^{1-\frac{1}{\beta}} \) for \( \frac{1}{2} \leq \beta < 1 \) and the constant \( 2^{1-\frac{1}{\beta}} \) is the best possible.

**Lemma 7.** ([5]) Let \( g(z) = z + b_2z^2 + b_3z^3 + \cdots \in S^* \), then \( |a_n| \leq n \) and \( |b_3 - \gamma b_2^2| \leq 2 \max\{1, |4\gamma - 3|\} \).

3. Some properties of functions in \( SB_\lambda(\beta) \)

**Theorem 2.**

\( SB_\lambda(\beta) \subset B(1 - 1/\lambda, \beta^{1/\lambda}) \).

**Proof.** Let \( f \in SB_\lambda(\beta) \). Then for some \( p \in P_\beta \) we have

\[
\frac{(zf'(z))^\lambda}{g(z)^{\lambda-1}f(z)} = \left( \frac{zf'(z)}{g(z)^{1-1/\lambda}f(z)^{1/\lambda}} \right)^\lambda = p(z)
\]

which implies

\[
\frac{zf'(z)}{g(z)^{1-1/\lambda}f(z)^{1/\lambda}} = p(z)^{1/\lambda}.
\]

So by Lemma 5, we have

\[
\Re \frac{zf'(z)}{g(z)^{1-1/\lambda}f(z)^{1/\lambda}} > \beta^{1/\lambda}.
\]

Taking \( \alpha = 1 - 1/\lambda \), we have \( f \in B(1 - 1/\lambda, \beta^{1/\lambda}) \). That is \( SB_\lambda(\beta) \subset B(1 - 1/\lambda, \beta^{1/\lambda}) \).

**Corollary 3.** All \( \lambda \)-pseudo-close-to-convex functions are Bazilević close-to-convex of type \( 1 - 1/\lambda \), order \( \beta^{1/\lambda} \) and univalent in \( E \).
**Theorem 4.** Let $f \in SB_\lambda(\beta)$. Then $f(z)$ has the integral representation

$$
f(z) = \begin{cases} 
\left( \frac{\lambda-1}{\lambda} \int_0^z \left( \frac{p(t)}{g(t)} \right)^{1/\lambda} \frac{g(t)}{t} \, dt \right)^{1/\lambda-1} & \text{if } \lambda > 1, \\
\exp \int_0^z \frac{p(t)}{t} \, dt & \text{if } \lambda = 1,
\end{cases}
$$

for some $p \in P_\beta$.

**Proof.** For $\lambda = 1$, the integral representation of starlike functions is well known. Now suppose $\lambda > 1$, then from Equation (1) we have

$$
\frac{zf'(z)}{g(z)^{1-1/\lambda} f(z)^{1/\lambda}} = p(z)^{1/\lambda}.
$$

Then taking $\alpha = 1 - 1/\lambda$, we have

$$
\frac{zf'(z)}{g(z)^{\alpha} f(z)^{1-\alpha}} = p(z)^{1-\alpha},
$$

so that

$$
(f(z)^\alpha)' = \frac{\alpha p(z)^{1-\alpha} g(z)^\alpha}{z}.
$$

Thus we have

$$
f(z) = \left( \alpha \int_0^z \frac{p(t)^{1-\alpha} g(t)^\alpha}{t} \, dt \right)^{1/\alpha}
$$

which, on replacing $\alpha = 1 - 1/\lambda$, we have the integral representation of $f \in SB_\lambda(\beta)$ for $\lambda > 1$. \qed

**Corollary 5.** Let $f \in SB_2(\beta)$. Then $f(z)$ has integral representation

$$
f(z) = \left( \frac{1}{2} \int_0^z (p(t)g(t))^{1/2} \frac{1}{t} \, dt \right)^2.
$$

**Theorem 6.** If $f \in A$ satisfies

$$
\text{Re} \left\{ \frac{\lambda z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > \frac{\beta(1 - 2\lambda) - 1}{2\beta}, \quad z \in E,
$$

then $f \in SB_\lambda(2^{1-1/\beta})$, $1/2 \leq \beta < 1$. The constant $2^{1-1/\beta}$ is the best possible.
Proof. For $z \in E$, define
\[ p(z) = \frac{(zf'(z))^{\lambda}}{g(z)^{\lambda-1} f(z)}. \]
Logarithmic differentiation gives
\[ \frac{p'(z)}{p(z)} = \frac{\lambda}{z} + \frac{\lambda f''(z)}{f'(z)} - (\lambda - 1) \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}. \]
Hence
\[ 1 + \frac{zp'(z)}{p(z)} = (\lambda + 1) + \frac{\lambda z f''(z)}{f(z)} - (\lambda - 1) \frac{z g'(z)}{g(z)} - \frac{z f'(z)}{f(z)} \]
so that
\[ \Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\beta - 1}{2\beta} \]
as in Lemma 6 implies (equivalently)
\[ \Re \left\{ \frac{\lambda z f''(z)}{f(z)} - \frac{z f'(z)}{f(z)} \right\} > \frac{\beta(1 - 2\lambda) - 1}{2\beta} \]
since $\Re z g'(z)/g(z) > 0$. Hence by Lemma 6, we have
\[ \Re \frac{(zf'(z))^{\lambda}}{g(z)^{\lambda-1} f(z)} > 2^{1-1/\beta}, \ 1/2 \leq \beta < 1 \]
as required. □

For $\lambda = 1$ and $\lambda = 1$, $\beta = 1/2$ we obtain the following corollaries.

**Corollary 7.** If $f \in A$ satisfies
\[ \Re \left\{ \frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right\} > -\frac{1 + \beta}{2\beta}, \ z \in E, \]
then $f \in S^*(2^{1-1/\beta})$, $1/2 \leq \beta < 1$. The constant $2^{1-1/\beta}$ is the best possible.

**Corollary 8.** If $f \in A$ satisfies
\[ \Re \left\{ \frac{\lambda z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right\} > -\frac{3}{2}, \ z \in E, \]
then $f \in S^*(1/2)$. The constant $1/2$ is the best possible.
Theorem 9. Let \( f \in SB_\lambda(\beta) \). Then

\[
|a_2| \leq \frac{2(\lambda - \beta)}{2\lambda - 1},
\]

\[
|a_3| \leq \begin{cases} 
\frac{2|F + (\lambda - 1)(12\lambda^2 - 14\lambda + 3)|}{2\lambda - 1}, & \text{if } 1 \leq \lambda \leq 1 + \frac{\sqrt{2}}{2}, \\
\frac{2(\lambda - \beta)(12\lambda^2 - 14\lambda + 3)}{(2\lambda - 1)^2(3\lambda - 1)}, & \text{if } \lambda \geq 1 + \frac{\sqrt{2}}{2},
\end{cases}
\]

where

\[
F = 2(1 - \beta)[4(2 + \beta)\lambda^2 - 2(3 + 4\beta)\lambda + 2\beta - 1].
\]

Proof. For \( f \in SB_\lambda(\beta) \), then there exists \( p \in P_\beta \) such that

\[
\frac{(zf'(z))^\lambda}{g(z)^{\lambda-1}f(z)} = p(z)
\]

so that

\[
(zf'(z))^\lambda = g(z)^{\lambda-1}f(z)p(z). \quad (2)
\]

By careful computation, the left hand side of (2) expands as:

\[
(zf'(z))^\lambda = z^\lambda \left( 1 + 2\lambda a_2 z + [3\lambda a_3 + 2\lambda(\lambda - 1)a_2^2]z^2 + \cdots \right)
\]

while its right hand side yields the series

\[
g(z)^{\lambda-1}f(z)p(z) = z^\lambda \left( 1 + [a_2 + c_1 + (\lambda - 1)b_2] z \\
+ \left[ a_3 + a_2 c_1 + c_2 + (\lambda - 1)(b_3 + a_2 b_2 + b_2 c_1 + \frac{\lambda - 2}{2} b_2^2) \right] z^2 + \cdots \right).
\]

Now comparing coefficients of both sides of (2), we have

\[
2a_2 \lambda = a_2 + (\lambda - 1)b_2 + c_1 \quad (3)
\]

from which we obtain

\[
a_2 = \frac{(\lambda - 1)b_2 + c_1}{2\lambda - 1}.
\]

Using Lemmas 3 and 7, we obtain the given bound on \( a_2 \). Also comparing coefficients further, we have

\[
3\lambda a_3 + 2\lambda(\lambda - 1)a_2^2 = a_3 + a_2 c_1 + c_2 + (\lambda - 1)(b_3 + a_2 b_2 + b_2 c_1 + \frac{\lambda - 2}{2} b_2^2)
\]
leading to

$$(3\lambda - 1)a_3 = a_2c_1 + c_2 + (\lambda - 1)(b_3 + a_2b_2 + b_2c_1 + \frac{\lambda - 2}{2}b_2^2 - 2\lambda(\lambda - 1)a_2^2).$$

Substituting $a_2$ and arranging, we have

$$(3\lambda - 1)a_3 = c_2 - \frac{2(2\lambda^2 - 4\lambda + 1)c_1^2}{(2\lambda - 1)^2}$$

$$+ (\lambda - 1)
\left(b_3 - \frac{\lambda}{2(2\lambda - 1)^2}b_2^2\right)
+ \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda - 1)^2}b_2c_1. \quad (4)$$

Noting that $4\lambda^2 - 5\lambda + 1$ is positive for $\lambda \geq 1$, we get

$$(3\lambda - 1)|a_3| \leq |c_2 - \frac{2(2\lambda^2 - 4\lambda + 1)c_1^2}{(2\lambda - 1)^2}|$$

$$+ (\lambda - 1)\left|b_3 - \frac{\lambda}{2(2\lambda - 1)^2}b_2^2\right| + \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda - 1)^2}|b_2||c_1|. $$

Now using Lemmas 3, 4 and 7, we obtain

$$(3\lambda - 1)|a_3| \leq 2(1 - \beta) \max \left\{1, \left|\frac{4\beta\lambda^2 + (4 - 8\beta)\lambda + 2\beta - 1}{(2\lambda - 1)^2}\right|\right\}$$

$$+ 2(\lambda - 1) \max \left\{1, \left|\frac{12\lambda^2 - 14\lambda + 3}{(2\lambda - 1)^2}\right|\right\} + 4(1 - \beta)\frac{4\lambda^2 - 5\lambda + 1}{(2\lambda - 1)^2}.$$ 

Furthermore, it is easy to see that

$$\max \left\{1, \left|\frac{12\lambda^2 - 14\lambda + 3}{(2\lambda - 1)^2}\right|\right\} = \frac{12\lambda^2 - 14\lambda + 3}{(2\lambda - 1)^2} \geq 1, \; \lambda \geq 1$$

and that

$$\max \left\{1, \left|\frac{4\beta\lambda^2 + (4 - 8\beta)\lambda + 2\beta - 1}{(2\lambda - 1)^2}\right|\right\} = \begin{cases} \frac{4\beta\lambda^2 + (4 - 8\beta)\lambda + 2\beta - 1}{(2\lambda - 1)^2}, & \text{if } 1 \leq \lambda \leq 1 + \frac{\sqrt{2}}{2} \\
1, & \text{if } \lambda \geq 1 + \frac{\sqrt{2}}{2} \end{cases}$$

so that if $1 \leq \lambda \leq 1 + \sqrt{2}/2$, then
\[(3\lambda - 1)|a_3| \leq 2(1 - \beta) \frac{4\beta \lambda^2 + (4 - 8\beta)\lambda + 2\beta - 1}{(2\lambda - 1)^2} + 2(\lambda - 1) \frac{12\lambda^2 - 14\lambda + 3}{(2\lambda - 1)^2} + 4(1 - \beta) \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda - 1)^2}\]

which on simplification gives the first part of the bound on \(a_3\), and if \(\lambda \geq 1 + \sqrt{2}/2\), then

\[(3\lambda - 1)|a_3| \leq 2(1 - \beta) + 2(\lambda - 1) \frac{12\lambda^2 - 14\lambda + 3}{(2\lambda - 1)^2} + 4(1 - \beta) \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda - 1)^2}\]

which also simplifies to the second part of the bound on \(a_3\) as desired. This completes the proof. \(\Box\)

If \(\beta = 0\), we have

**Corollary 10.** Let \(f \in SB_{\lambda}\). Then

\[|a_2| \leq \frac{2\lambda}{(2\lambda - 1)},\]

\[|a_3| \leq \begin{cases} \frac{2(12\lambda^3 - 18\lambda^2 + 11\lambda - 2)}{(2\lambda - 1)^2(3\lambda - 1)}, & \text{if } 1 \leq \lambda \leq 1 + \frac{\sqrt{2}}{2}, \\ 2\lambda(12\lambda^2 - 14\lambda + 3)}{(2\lambda - 1)^2(3\lambda - 1)}, & \text{if } \lambda \geq 1 + \frac{\sqrt{2}}{2}. \end{cases}\]

**Theorem 11.** Let \(f \in SB_{\lambda}(\beta), \lambda > 1\). For real number \(\mu \leq (4\lambda^2 - 5\lambda + 1)/(2(\lambda - 1)(3\lambda - 1))\)

\[|a_3 - \mu a_2^2| \leq \frac{2(1 - \beta)G}{3\lambda - 1} + \frac{(\lambda - 1)H}{3\lambda - 1} + \frac{4(1 - \beta)[4\lambda^2 - 5\lambda - 2\mu(\lambda - 1)(3\lambda - 1) + 1]}{(2\lambda - 1)^2(3\lambda - 1)},\]

where

\[G = \max \left\{ 1, \left| \frac{4\beta \lambda^2 + (4 - 8\beta)\lambda - 2(1 - \beta)\mu(3\lambda - 1) + 2\beta - 1}{(2\lambda - 1)^2} \right| \right\}\]

and

\[H = \max \left\{ 1, \left| \frac{12\lambda^2 - 14\lambda - 4\mu(\lambda - 1)(3\lambda - 1) + 3}{(2\lambda - 1)^2} \right| \right\}.\]
Proof. Using Equations (3) and (4), and arranging, we find
\[
a_3 - \mu a_2^2 = \frac{1}{3\lambda - 1} \left[ c^2 - \frac{2(2\lambda^2 - 4\lambda + \mu(3\lambda - 1) + 1) c_1^2}{(2\lambda - 1)^2} \right] \\
+ \frac{\lambda - 1}{3\lambda - 1} \left[ b_3 - \frac{\lambda + 2\mu(\lambda - 1)(3\lambda - 1) b_2^2}{2(2\lambda - 1)^2} \right] \\
+ \frac{4\lambda^2 - 5\lambda - 2\mu(\lambda - 1)(3\lambda - 1) + 1}{(2\lambda - 1)^2} b_2 c_1.
\]
Hence the inequalities follow using Lemmas 3, 4 and 7, and the condition on \(\mu\). For \(\lambda = 1\), the Fekete-Szego inequalities for the special cases \(SB_1(\beta) \equiv S^*(\beta)\) are well known.

Finally we give two nontrivial examples of pseudo-close-to-convex functions for \(\lambda = 2\).

**Example.** The functions \(f_j(z), j = 1, 2\) given by
\[
f_j(z) = \begin{cases} 
(tanh^{-1}\sqrt{z})^2 = \left(\frac{1}{2} \ln \frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2 = z + \cdots & \text{if } j = 1, \\
(tan^{-1}\sqrt{z})^2 = z + \cdots & \text{if } j = 2,
\end{cases}
\]
both belong to \(SB_2\).

**Proof.** For each \(f_j(z)\), take
\[
g_j(z) = \begin{cases} 
z/(1 - z) & \text{if } j = 1, \\
z/(1 + z) & \text{if } j = 2.
\end{cases}
\]
Then by careful computation we find that
\[
\frac{(zf_j'(z))^2}{g_j(z)f_j(z)} = \begin{cases}
1/(1 - z) & \text{if } j = 1, \\
1/(1 + z) & \text{if } j = 2.
\end{cases}
\]
Since the right hand side of both equations above are all functions in \(P\), we have
\[
Re \frac{(zf_j'(z))^2}{g_j(z)f_j(z)} > 0, \quad j = 1, 2,
\]
so \(f_j \in SB_2\). The proof is complete.

Finally, we remark that given the definition of the new class \(SB_\lambda(\beta)\), the two examples above point to the univalence in the unit disk of some transcendental functions under certain geometric conditions.
References


