STABILITY OF A TIMOSHENKO SYSTEM
WITH CONSTANT DELAY

Innocent Ouedraogo\textsuperscript{1}§, Gilbert Bayili\textsuperscript{2}

\textsuperscript{1,2} Department of Mathematics
Joseph KI-ZERBO University
03 BP 7021 Ouagadougou, BURKINA FASO

Abstract: The aim of this work is to develop a detail analysis of a Timoshenko type beam model taking into account a delay. We prove the well-posedness and regularity of solution, explained using the theory of the Faedo-Galerkin scheme. Namely, under a suitable choice of Lyapunov functional, exponential decay of the whole energy holds.

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1. Introduction

Here we consider the following Shear beam model and new facts related to the classical Timoshenko system with internal delay

\begin{equation}
\begin{aligned}
\rho_1 \varphi_{tt}(x,t) - \kappa (\varphi_x(x,t) + \psi(x,t))_x + \mu_0 \varphi_t(x,t) \\
+ \mu_1 \varphi_t(x,t - \tau) = 0 \quad \text{in } ]0,L[ \times (0, +\infty), \\
- b \psi_{xx}(x,t) + \kappa (\varphi_x(x,t) + \psi(x,t)) = 0 \quad \text{in } ]0,L[ \times (0, +\infty).
\end{aligned}
\end{equation}

Additionally, we consider initial conditions given by

\[ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad x \in ]0,L[, \]

where \( \varphi_0, \varphi_1, \psi_0 \) are given functions, and the boundary conditions of Dirichlet

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§Correspondence author
are given by
\[ \psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0. \]

Note that the functions \( \varphi \) and \( \psi \) describe the transverse displacement of the beam and the rotation angle of a filament of the beam, respectively. \( \rho_1, \kappa, \mu_0, \mu_1, b \) and \( \tau \) are positive constants.

It is well known that if \( \mu_1 = 0 \), that is in absence of delay, the corresponding following system:

\[
\begin{cases}
\rho_1 \varphi_{tt}(x,t) - \kappa (\varphi_x(x,t) + \psi(x,t)) + \mu \varphi_t(x,t) = 0 & \text{in } ]0,L[ \times (0, +\infty), \\
-b \psi_{xx}(x,t) + \kappa (\varphi_x(x,t) + \psi(x,t)) = 0 & \text{in } ]0,L[ \times (0, +\infty), \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x) & \text{in } ]0,L[, \\
\psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0,
\end{cases}
\]

was analyzed in a recent paper by Almeida and al. [1], where it was shown that the energy of the system decays exponentially to zero.

In the case of the wave equations, Nicaise and al. [8] investigated exponential stability results with delay concentrated at \( \tau \) for the system

\[
\begin{cases}
u_{tt}(x,t) - \Delta u(x,t) = 0 & \text{in } \Omega \times (0, +\infty), \\
u(x,t) = 0 & \text{on } \Gamma_D \times (0, +\infty), \\
\frac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t - \tau) = 0 & \text{on } \Gamma_N \times (0, +\infty), \\
u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{in } \Omega, \\
u(x,t - \tau) = f_0(x,t - \tau) & \text{on } \Gamma_N \times (0, \tau),
\end{cases}
\]

under the condition \( \mu_2 < \mu_1 \), by combining inequalities due to Carleman estimates and compactness-uniqueness arguments.

Recently Bayili and al. [2] in the case of the wave equation with dynamical control, studied a wave equation set in a bounded domain with a dynamical control. For the strong stability result, they use the spectral decomposition theory of Sz-Nagy-Foias Parrott [9], Claude, Chen and al [5],[3]. Then they
prove that if the delay term is small enough, then the system with delay

\[
\begin{cases}
u_t(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0, +\infty), \\
u(x,t) = 0 \text{ on } \Gamma_D \times (0, +\infty), \\
\frac{\partial u}{\partial n}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0, +\infty), \\
\eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau) = 0 \\
\text{ on } \Gamma_N \times (0, +\infty), \\
u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega, \\
\eta(x,0) = \eta_0 \text{ on } \Gamma_N, \\
\eta(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N \times (0, \tau),
\end{cases}
\]

has the same (polynomial) decay rate than the one without delay. In this paper, we assume that there exists a positive constant \(\zeta\) verifying

\[
\tau \mu_0 < \zeta < \tau(2\mu_0 - \mu_1).
\]

The remaining of the paper is organized as follows. In Section 2 the well-posedness of the problem (1) is analyzed using the Faedo-Galerkin method. And finally in Section 3 we prove the exponential decay of the energy when time goes to infinity using a Lyapounov function.

### 2. Well-posedness of the problem

In this section we will give well-posedness results for problem (1) using Faedo-Galerkin method. To this aim, we introducing the following auxiliary change of variable

\[
z(x,\rho,t) = \varphi_t(x,t-\tau \rho), \quad x \in ]0,L[, \quad \rho \in (0,1), \quad t > 0.
\]

The problem (1) is now equivalent to

\[
\begin{cases}
\rho_1 \varphi_{tt}(x,t) - \kappa(\varphi_x(x,t) + \psi(x,t)) + \mu_0 \varphi_t(x,t) + \mu_1 z(x,1,t) = 0 \\
\text{ in } ]0,L[ \times (0, +\infty), \\
-\psi_{xx}(x,t) + \kappa(\varphi_x(x,t) + \psi(x,t)) = 0 \quad \text{ in } ]0,L[ \times (0, +\infty), \\
\tau z_t(\rho,t) + z_\rho(\rho,t) = 0 \quad \text{ in } (0,1) \times (0, +\infty); \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x) \quad \text{ in } ]0,L[, \\
z(x,\rho,0) = f_0(x,-\rho \tau) \quad \forall \ \rho \in (0,1), \\
z(x,0,t) = \varphi_t(x,t) \quad \forall t \in (0, +\infty) \\
\psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0.
\end{cases}
\]
Let us consider the Hilbert spaces
\[ \mathcal{H} = H^1(0, L) \times L^2(0, L) \times H^1_*(0, L) \times L^2_*(0, L), \]
and
\[ \mathcal{H}_1 = \left( H^2(0, L) \cap H^1_0(0, L) \right)^2 \times H^2_*(0, L) \times H^1(0, L) \times (0, 1), \]
where
\[ L^2_*(0, L) = \left\{ u \in L^2(0, L), \int_0^L u(x)dx = 0 \right\}, \]
\[ H^1_*(0, L) = H^1(0, L) \cap L^2_*(0, L), \]
and
\[ H^2_*(0, L) = H^2(0, L) \cap H^1_*(0, L). \]
We equipped \( \mathcal{H} \) with the norm
\[
\left\| (u, v, w, z)^T \right\|_{\mathcal{H}}^2 = \frac{\rho_1}{2} \int_0^L |v|^2 dx + \frac{b}{2} \int_0^L |w_x|^2 dx
+ \frac{\kappa}{2} \int_0^L |u_x + w|^2 dx + \frac{\zeta}{2} \int_0^L \int_0^1 |z|^2 d\rho dx,
\]
where \( \zeta \) is a positive constant verifying
\[ \tau \mu_0 < \zeta < \tau (2 \mu_0 - \mu_1). \tag{8} \]
Let \((\varphi, \psi, z)\) be a solution of (7), the corresponding energy is given by
\[
E(t) = \frac{\rho_1}{2} \int_0^L |\varphi_t|^2 dx + \frac{b}{2} \int_0^L |\psi|^2 dx + \frac{\kappa}{2} \int_0^L |\varphi_x + \psi|^2 dx
+ \frac{\zeta}{2} \int_0^L \int_0^1 |z|^2 d\rho dx.
\]

**Theorem 1.** Under the assumption (5) we have
\[
\frac{d}{dt} E(t) \leq 0.
\]

**Proof.** Multiplying (7)_1 by \( \varphi_t \) and integrating on \([0; L]\), we have
\[
\rho_1 \int_0^L \varphi_{tt}(x,t)\varphi_t(x,t)dx - \kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]_x \varphi_t(x,t)dx
\]
\[ + \mu_0 \int_0^L \varphi_t(x,t) \varphi_t(x,t) \, dx + \mu_1 \int_0^L z(x,1,t) \varphi_t(x,t) \, dx = 0. \]

Integrating by parts, we obtain
\[
\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi_t(x,t)|^2 \, dx - \kappa \left[ (\varphi_x + \psi) \varphi_t(x,t) \right]_0^L \\
+ \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \varphi_{tx}(x,t) \, dx + \mu_0 \int_0^L |\varphi_t(x,t)|^2 \, dx \\
+ \mu_1 \int_0^L z(x,1,t) \varphi_t(x,t) \, dx = 0.
\]

Since \( \varphi \) is zero at 0 and \( L \), we have
\[
\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi_t(x,t)|^2 \, dx + \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \varphi_t(x,t) \, dx \\
+ \mu_0 \int_0^L |\varphi_t(x,t)|^2 \, dx + \mu_1 \int_0^L z(x,1,t) \varphi_t(x,t) \, dx = 0. \quad (9)
\]

Multiply (7)_2 by \( \psi_t \) and integrate on \([0; L]\) we have
\[
- b \int_0^L \psi_{xx}(x,t) \psi_t(x,t) \, dx + \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \psi_t(x,t) \, dx = 0.
\]

Integrating by parts, we obtain
\[
- b \left[ \psi_x \psi_t \right]_0^L + b \int_0^L \psi_x(x,t) \psi_{xt}(x,t) \, dx \\
+ \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \psi_t(x,t) \, dx = 0.
\]

Since \( \psi_x \) is zero at 0 and \( L \), we have
\[
b \int_0^L \psi_x(x,t) \psi_{xt}(x,t) \, dx + \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \psi_t(x,t) \, dx = 0,
\]

which is written as
\[
b \frac{d}{dt} \int_0^L |\psi_x(x,t)|^2 \, dx + \kappa \int_0^L \left[ \varphi_x(x,t) + \psi(x,t) \right] \psi_t(x,t) \, dx = 0. \quad (10)
\]
Multiply (7) by $\frac{\zeta}{\tau}z$ and integrate on $[0; L] \times [0; 1]$ we have

$$\zeta \int_0^L \int_0^1 z_t(x, \rho, t)z(x, \rho, t) d\rho dx + \tau \int_0^L \int_0^1 z(\rho(x, \rho, t))z(x, \rho, t) dx = 0,$$

written as

$$\frac{\zeta}{\tau} \frac{d}{dt} \int_0^L \int_0^1 |z(x, \rho, t)|^2 d\rho dx + \frac{\zeta}{\tau} \int_0^L \left[ |z(x, \rho, t)|^2 \right]_0^1 dx = 0.$$

Finally,

$$\frac{\zeta}{\tau} \frac{d}{dt} \int_0^L \int_0^1 \left[ |z(x, \rho, t)|^2 \right]_0^1 dx = 0. \quad (11)$$

By adding (9) – (11) we obtain

$$\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi_t(x, t)|^2 dx + \kappa \int_0^L [\varphi_x(x, t) + \psi(x, t)] \varphi_{tx}(x, t) dx$$

$$+ \mu_0 \int_0^L |\varphi_t(x, t)|^2 dx + \mu_1 \int_0^L z(x, 1, t)\varphi_t(x, t) dx$$

$$+ \frac{b}{2} \frac{d}{dt} \int_0^L |\psi_x(x, t)|^2 dx + \kappa \int_0^L [\varphi_x(x, t) + \psi(x, t)] \psi_t(x, t) dx$$

$$+ \mu_1 \int_0^L z(x, 1, t)\varphi_t(x, t) dx + \frac{b}{2} \frac{d}{dt} \int_0^L |\psi_x(x, t)|^2 dx$$

$$+ \frac{\zeta}{2\tau} \frac{d}{dt} \int_0^L \int_0^1 \left[ |z(x, \rho, t)|^2 \right]_0^1 d\rho dx$$

This means that

$$\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi_t(x, t)|^2 dx + \mu_0 \int_0^L |\varphi_t(x, t)|^2 dx$$

$$+ \kappa \int_0^L [\varphi_x(x, t) + \psi(x, t)] [\varphi_x(x, t) + \psi(x, t)] dx$$

$$+ \mu_1 \int_0^L z(x, 1, t)\varphi_t(x, t) dx + \frac{b}{2} \frac{d}{dt} \int_0^L |\psi_x(x, t)|^2 dx$$

$$+ \frac{\zeta}{2\tau} \frac{d}{dt} \int_0^L \int_0^1 \left[ |z(x, \rho, t)|^2 \right]_0^1 d\rho dx$$
\[ + \frac{\zeta}{2\tau} \int_0^L \left[ |z(x, 1, t)|^2 - |z(x, 0, t)|^2 \right] \, dx = 0. \]

So,

\[
\frac{d}{dt} \left\{ \frac{\rho_1}{2} \int_0^L |\varphi_t(x, t)|^2 \, dx + \frac{\kappa}{2} \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 \, dx \right. \\
+ \frac{b}{2} \int_0^L |\psi_x(x, t)|^2 \, dx + \frac{\zeta}{2} \int_0^L \int_0^1 |z(x, \rho, t)|^2 \, d\rho \, dx \left. \right\} \\
+ \mu_0 \int_0^L |\varphi_t(x, t)|^2 \, dx + \mu_1 \int_0^L z(x, 1, t)\varphi_t(x, t) \, dx \\
+ \frac{\zeta}{2\tau} \int_0^L \left[ |z(x, 1, t)|^2 - |z(x, 0, t)|^2 \right] \, dx = 0. \]

And this leads to

\[
d \frac{d}{dt} E(t) = -\mu_0 \int_0^L |\varphi_t(x, t)|^2 \, dx - \mu_1 \int_0^L z(x, 1, t)\varphi_t(x, t) \, dx \\
- \frac{\zeta}{2\tau} \int_0^L \left[ |z(x, 1, t)|^2 - |z(x, 0, t)|^2 \right] \, dx. \quad (12)
\]

We also know that

\[
z(x, 1, t)z(x, 0, t) = \frac{1}{2} |z(x, 1, t) + z(x, 0, t)|^2 - \frac{1}{2} |z(x, 0, t)|^2 \\
- \frac{1}{2} |z(x, 1, t)|^2. \quad (13)
\]

Using (13) in (12), we have

\[
\frac{d}{dt} E(t) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^L |z(x, 0, t)|^2 \, dx \\
- \frac{\mu_1}{2} \int_0^L |z(x, 1, t) + z(x, 0, t)|^2 \, dx
\]
We have \(-\frac{\mu_1}{2} < 0\) and by hypothesis (5), we get that \(\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} < 0\) and \(\frac{\tau\mu_1 - \zeta}{2\tau} < 0\).

Therefore \(\frac{d}{dt}E(t) \leq 0\), hence the system (7) is dissipative.

3. Existence and uniqueness

In order to state our main result in this section, we start by defining that we mean by a weak solution of the problem (7) as follows:

**Definition 2.** Given an initial data \(U_0 = (\varphi_0, \varphi_1, \psi_0, z_0) \in \mathcal{H}\), a function \(U = (\varphi, \varphi_t, \psi, z) \in C([0, T]; \mathcal{H})\) is said to be weak solution of (7) if for almost everywhere \(t \in [0, T]\),

\[
\begin{align*}
\rho_1 \frac{d}{dt}(\varphi_t, u) + \kappa(\varphi_x(x, t) + \psi(x, t), u_x) + \mu_0(\varphi_t, u) + \mu_1(z(., 1, t), u) &= 0, \\
b(\psi_x, v_x) + \kappa(\varphi_x + \psi, v) &= 0, \\
\tau(z_t, w) + (z, w) &= 0,
\end{align*}
\]

for all \(u \in H_1^0(0, L), w \in H_2^2(0, L), \psi \in H_1^1(0, L)\times(0, 1)\) and \((\varphi(0), \varphi_t(0), \psi(0), z(0)) = (\varphi_0, \varphi_1, \psi_0, z_0)\).

The main result of this section is the following:

**Theorem 3.** Assume that (5) holds. Then for any data \(U_0 = (\varphi_0, \varphi_1, \psi_0, f_0) \in \mathcal{H}\), the problem (7) has one and only one weak solution \(U = (\varphi, \varphi_t, \psi, z)\) verifying:

\[
\begin{align*}
\varphi &\in L^\infty(0, T, H_0^1(0, L)), \\
\varphi_t &\in L^\infty(0, T, L^2(0, L)), \\
\psi &\in L^\infty(0, T, H_1^1(0, L)), \\
z &\in L^\infty(0, T, L^2((0, L) \times (0, 1)))
\end{align*}
\]
Moreover, if $U_0 = (\varphi_0, \varphi_1, \psi_0, f_0)$ belongs to $\mathcal{H}_1$, then problem (7) has one and only one strong weak solution $U = (\varphi, \varphi_t, \psi, z)$ which satisfies

$$
\begin{aligned}
\varphi &\in L^\infty(0, T, H^2(0, L) \cap H^1_0(0, L)), \\
\varphi_t &\in L^\infty(0, T, H^2_0(0, L)), \\
\psi &\in L^\infty(0, T, H^2_0(0, L)), \\
z &\in L^\infty(0, T, H^1((0, L) \times (0, 1))).
\end{aligned}
$$

(17)

**Proof.** The Faedo-Galerkin method will be the key to prove the existence of a global solution.

**Step 1.** Let us consider initial data $(\varphi_0, \varphi_1, \psi_0, f_0) \in \mathcal{H}$.  

Let $\{u^k\}, k \in \mathbb{N}^*$ and $\{v^k\}, k \in \mathbb{N}^*$ basics formed by eigenfunctions of $-\partial_{xx}$. This bases can be considered orthogonal in $H^2(\Omega) \cap H^1_0(\Omega)$ and $H^2_0(0, L)$ respectively, and both orthonormal in $L^2(0; L)$.

As Yazid, Chen et al. in [10],[4], we also define the sequence $\{w^k\}, k \in \mathbb{N}^*$ in the following way

$$w^k(x, 0) = u^k(x) \quad \text{then we extend } w^k(x, 0) \text{ by } w^k(x, \rho)$$

on $L^2((0, L) \times (0, 1))$.

Approximation spaces $H_n, V_n$ and $W_n$ of finite dimensions are given by

$$H_n = \text{span} \left\{ u^1, u^2, ..., u^n \right\}, \quad V_n = \text{span} \left\{ v^1, v^2, ..., v^n \right\}$$

and $W_n = \text{span} \left\{ w^1, w^2, ..., w^n \right\}, n \in \mathbb{N}^*$.

We will find an approximate solution of the form:

$$\varphi^n(t, x) = \sum_{j=1}^{n} a^{jn}(t) u^j(x),$$

$$\psi^n(t, x) = \sum_{j=1}^{n} b^{jn}(t) v^j(x),$$

$$z^n(x, t, \rho) = \sum_{j=1}^{n} c^{jn}(t) w^j(x, \rho),$$
to the following approximate problem

\[
\begin{align*}
\left\{ \begin{aligned}
\rho_1 \int_0^L \varphi^n_{tt}(x,t)u\,dx - \kappa \int_0^L [\varphi^n_x(x,t) + \psi^n(x,t)]_x u\,dx \\
+ \mu_0 \int_0^L \varphi^n_t(x,t)u\,dx + \mu_1 \int_0^L z^n(x,1,t)u\,dx = 0, \\
-b \int_0^L \psi^n_{xx}(x,t)v\,dx + \kappa \int_0^L [\varphi^n_x(x,t) + \psi^n(x,t)] v\,dx = 0, \\
\zeta \int_0^L \int_0^1 z^n_t(x,\rho,t)w\,d\rho\,dx + \frac{\zeta}{\tau} \int_0^L \int_0^1 z^n_{\rho}(x,\rho,t)w\,d\rho\,dx = 0,
\end{aligned} \right.
\end{align*}
\]  

(18)

for all \( u \in H_n, v \in V_n, w \in W_n \), with initial conditions such that

\[
(\varphi^n(0), \varphi^n_t(0), \psi^n(0), z^n(0)) = (\varphi^n_0, \varphi^n_1, \psi^n_0, z^n_0) \to (\varphi_0, \varphi_1, \psi_0, f_0),
\]

strongly in \( \mathcal{H} \).

Note that \( a^{j_n}, b^{j_n} \) and \( c^{j_n}, 1 \leq j \leq n \) form the temporal weighting coefficients.

According to the standard theory of ordinary differential equations, the finite dimensional problem (18) – (19) has a solution \((a^{j_n}, b^{j_n}, c^{j_n}), 1 \leq j \leq n \) defined on \([0, t_n]\) for every \( n \in \mathbb{N}^* \).

Then the a priori estimates that follow imply that in fact \( t_n = T, \forall T > 0 \).

**Step 2. A Priori Estimate I**

Replacing \( u \) by \( \varphi^n_t \) in (18)\(_1\), \( v \) by \( \psi^n_t \) in (18)\(_2\) and \( w \) by \( z^n \) in (18)\(_3\), we obtain

\[
\begin{align*}
\left( \begin{aligned}
\rho_1 \frac{d}{dt} \int_0^L |\varphi^n_t(x,t)|^2\,dx + \kappa \int_0^L [\varphi^n_x(x,t) + \psi^n(x,t)] \varphi^n_{tx}(x,t)\,dx \\
+ \mu_0 \int_0^L |\varphi^n_t(x,t)|^2\,dx + \mu_1 \int_0^L z^n(x,1,t)\varphi^n_t(x,t)\,dx = 0, \\
-b \int_0^L \psi^n_{xx}(x,t)v\,dx + \kappa \int_0^L [\varphi^n_x(x,t) + \psi^n(x,t)] v\,dx = 0, \\
\zeta \int_0^L \int_0^1 z^n_t(x,\rho,t)w\,d\rho\,dx + \frac{\zeta}{\tau} \int_0^L \int_0^1 z^n_{\rho}(x,\rho,t)w\,d\rho\,dx = 0.
\end{aligned} \right.
\]

(20)

By making the same transformations as in the session of the dissipative character, we obtain

\[
\frac{d}{dt} E^n(t) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^L |z^n(x,0,t)|^2\,dx
\]
\[- \frac{\mu_1}{2} \int_0^L |z^n(x, 1, t) + z^n(x, 0, t)|^2 \, dx
\]
\[+ \left[ \frac{\tau \mu_1 - \zeta}{2 \tau} \right] \int_0^L |z^n(x, 1, t)|^2 \, dx, \quad (21)\]

where

\[
E^n(t) = \frac{\rho_1}{2} \int_0^L |\varphi^n_t|^2 \, dx + \frac{b}{2} \int_0^L |\psi^n_x|^2 \, dx + \frac{\kappa}{2} \int_0^L |\varphi^n_x + \psi^n|^2 \, dx
\]
\[+ \frac{\zeta}{2} \int_0^L \int_0^1 |z^n|^2 \, d\rho \, dx.\]

Thus integrating (21) from 0 to \(t < t_n\), we obtain from our choice of initial data that for all \(t \in [0; t_n]\) and for every \(n \in \mathbb{N}^*\),

\[
E^n(t) - E^n(0) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^t \int_0^L |z^n(x, 0, s)|^2 \, dx \, ds
\]
\[+ \left[ \frac{\tau \mu_1 - \zeta}{2 \tau} \right] \int_0^t \int_0^L |z^n(x, 1, s)|^2 \, dx \, ds
\]
\[= \frac{\mu_1}{2} \int_0^t \int_0^L |z^n(x, 1, t) + z^n(x, 0, s)|^2 \, dx \, ds
\]
which means

\[
E^n(t) = E^n(0)
\]

where

\[
E^n(0) = \frac{\rho_1}{2} \int_0^L |\varphi^n_t(x, 0)|^2 \, dx + \frac{b}{2} \int_0^L |\psi^n_x(x, 0)|^2 \, dx
\]
\[+ \frac{\kappa}{2} \int_0^L |\varphi^n_x(x, 0) + \psi^n(x, 0)|^2 \, dx
\]
\[+ \frac{\zeta}{2} \int_0^L \int_0^1 |z^n(x, \rho, 0)|^2 \, d\rho \, dx.\]
As \((\varphi^n(0), \varphi^n_1(0), \psi^n(0), z^n(0)) = (\varphi^n_0, \varphi^n_1, \psi^n_0, z^n_0) \to (\varphi_0, \varphi_1, \psi_0, f_0)\) strongly in \(\mathcal{H}\), then there exists a positive constant \(C_1\) such that \(E^n(0) \leq C_1\). Hence

\[
E^n(t) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^t \int_0^L |z^n(x,0,s)|^2 dx ds \\
- \left[ \frac{\tau \mu_1 - \zeta}{2\tau} \right] \int_0^t \int_0^L |z^n(x,1,s) + z^n(x,0,s)|^2 ds dx \\
- \frac{\mu_1}{2} \int_0^t \int_0^L |z^n(x,1,s) + z^n(x,0,s)|^2 ds dx < C_1,
\]

which means that

\[
\frac{\rho_1}{2} \int_0^L |\varphi^n_t|^2 dx + \frac{b}{2} \int_0^L |\psi^n_x|^2 dx + \frac{\kappa}{2} \int_0^L |\varphi^n_x + \psi^n|^2 dx \\
+ \frac{\zeta}{2} \int_0^L \int_0^1 |z^n|^2 dp dx \\
- \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^t \int_0^L |z^n(x,0,s)|^2 dx ds \\
+ \frac{\mu_1}{2} \int_0^t \int_0^L |z^n(x,1,s) + z^n(x,0,s)|^2 ds dx \\
- \left[ \frac{\tau \mu_1 - \zeta}{2\tau} \right] \int_0^t \int_0^L |z^n(x,1,s)|^2 ds dx \\
\leq C_1.
\]

As the constant \(C_1\) does not depend on \(n\), we can therefore take \(t_n = T\), for all \(T > 0\).

**Step 3. A Priori Estimate II**

Let us derive the equation \((18)_1\) with respect to \(t\) and then replacing \(u\) by \(\varphi^n_{tt}\). We obtain

\[
\rho_1 \int_0^L \varphi^n_{ttt}(x,t) \varphi^n_{tt}(x,t) dx - \kappa \int_0^L [\varphi^n_{xt}(x,t) + \psi^n_t(x,t)]_x \varphi^n_{tt}(x,t) dx \\
+ \mu_0 \int_0^L \varphi^n_{tt}(x,t) \varphi^n_{tt}(x,t) dx + \mu_1 \int_0^L z^n_t(x,1,t) \varphi^n_{tt}(x,t) dx = 0.
\]

By integrating by parts, we obtain

\[
\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi^n_{tt}(x,t)|^2 dx - \kappa [(\varphi^n_{xt} + \psi^n_t)\varphi^n_{tt}(x,t)]_0^t = 0.
\]
\begin{align*}
+ \kappa \int_0^L \left( \varphi^n_{xt}(x, t) + \psi^n_t(x, t) \right) \varphi^n_{xtt}(x, t) dx \\
+ \mu_0 \int_0^L |\varphi^n_{tt}(x, t)|^2 dx + \mu_1 \int_0^L z^n_i(x, 1, t) \varphi^n_{ii}(x, t) dx = 0.
\end{align*}

As \( \varphi \) is null at 0 and at \( L \), we have

\begin{align*}
\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\varphi^n_{tt}(x, t)|^2 dx + \kappa \int_0^L \left[ \varphi^n_{x}(x, t) + \psi^n_t(x, t) \right] \varphi^n_{xtt}(x, t) dx \\
+ \mu_0 \int_0^L |\varphi^n_{tt}(x, t)|^2 dx + \mu_1 \int_0^L z^n_i(x, 1, t) \varphi^n_{ii}(x, t) dx = 0. \tag{23}
\end{align*}

Let us derive the equation (18)\(_2\) with respect to \( t \), and then replacing \( v \) by \( \psi^n_{tt} \), we obtain

\begin{align*}
- b \int_0^L \psi^n_{xxt}(x, t) \psi^n_{tt}(x, t) dx \\
+ \kappa \int_0^L \left[ \varphi^n_{x}(x, t) + \psi^n_t(x, t) \right] \psi^n_{tt}(x, t) dx = 0.
\end{align*}

Integrating by parts, we obtain

\begin{align*}
- b \left[ \psi^n_{xt}(x, t) \psi^n_{tt}(x, t) \right]_0^L + b \int_0^L \psi^n_{x}(x, t) \psi^n_{xtt}(x, t) dx \\
+ \kappa \int_0^L \left[ \varphi^n_{x}(x, t) + \psi^n_t(x, t) \right] \psi^n_{tt}(x, t) dx = 0.
\end{align*}

Since \( \psi_x \) is zero at the edge we have

\begin{align*}
b \int_0^L \psi^n_{x}(x, t) \psi^n_{xtt}(x, t) dx \\
+ \kappa \int_0^L \left[ \varphi^n_{x}(x, t) + \psi^n_t(x, t) \right] \psi^n_{tt}(x, t) dx = 0.
\end{align*}

This is written as

\begin{align*}
\frac{b}{2} \frac{d}{dt} \int_0^L |\psi^n_{x}(x, t)|^2 dx \\
+ \kappa \int_0^L \left[ \varphi^n_{x}(x, t) + \psi^n_t(x, t) \right] \psi^n_{tt}(x, t) dx = 0. \tag{24}
\end{align*}
Summing (23) and (24), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\{ \rho_1 \int_0^L |\varphi_{tt}^n(x,t)|^2 dx + k \int_0^L |\varphi_{xt}^n(x,t) + \psi_t^n(x,t)|^2 dx \\
+ b \int_0^L |\psi_{xt}^n(x,t)|^2 dx \right\} + \mu_0 \int_0^L |\varphi_{tt}^n(x,t)|^2 dx \\
+ \mu_1 \int_0^L z_t^n(x,1,t) \varphi_{tt}^n(x,t) \, dx = 0,
\]
which means that
\[
\frac{d}{dt} G^n(t) + \mu_1 \int_0^L z_t^n(x,1,t) \varphi_{tt}^n(x,t) \, dx \\
+ \mu_0 \int_0^L |\varphi_{tt}^n(x,t)|^2 dx = 0,
\]
(25)
where
\[
G^n(t) = \frac{\rho_1}{2} \int_0^L |\varphi_{tt}^n(x,t)|^2 dx + \frac{k}{2} \int_0^L |\varphi_{xt}^n(x,t) + \psi_t^n(x,t)|^2 dx \\
+ \frac{b}{2} \int_0^L |\psi_{xt}^n(x,t)|^2 dx.
\]
By integrating (25) from 0 to \(t\) we have
\[
G^n(t) + \mu_1 \int_0^t \int_0^L z_s^n(x,1,s) \varphi_{ss}^n(x,s) \, dx \, ds \\
+ \mu_0 \int_0^t \int_0^L |\varphi_{ss}^n(x,s)|^2 \, dx \, ds = G^n(0),
\]
where
\[
G^n(0) = \frac{\rho_1}{2} \int_0^L |\varphi_{tt}^n(x,0)|^2 dx + \frac{k}{2} \int_0^L |\varphi_{xt}^n(x,0) + \psi_t^n(x,0)|^2 dx \\
+ \frac{b}{2} \int_0^L |\psi_{xt}^n(x,0)|^2 dx.
\]
As \((\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)\) strongly in \(H\), then there exists a positive constant \(C_2\) such that \(G^n(0) \leq C_2\). Hence
\[
G^n(t) + \mu_1 \int_0^t \int_0^L z_s^n(x,1,s) \varphi_{ss}^n(x,s) \, dx \, ds \\
+ \mu_0 \int_0^t \int_0^L |\varphi_{ss}^n(x,s)|^2 \, dx \, ds \leq C_2.
\]
(26)
Step 4. Passage to Limit

From (22) and (26) we have that:

\[
\begin{align*}
\{\varphi^n\} & \text{ is bounded in } L^\infty(0, T, H^1_0(0, L)), \\
\{\varphi^n_t\} & \text{ is bounded in } L^\infty(0, T, L^2(0, L)), \\
\{\varphi^n_{tt}\} & \text{ is bounded in } L^\infty(0, T, L^2(0, L)), \\
\{\psi^n\} & \text{ is bounded in } L^\infty(0, T, H^1_+(0, L)), \\
\{\psi^n_t\} & \text{ is bounded in } L^\infty(0, T, L^2(0, L)), \\
\{z^n\} & \text{ is bounded in } L^\infty(0, T, L^2((0, L) \times (0, 1))).
\end{align*}
\]

So we can extract subsequences \(\{\varphi^n\}, \{\psi^n\}\) and \(\{z^n\}\) such as

\[
\{\varphi^n\} \to \varphi \text{ weakly star in } L^\infty(0, T, H^1_0(0, L)),
\]

\[
\{\varphi^n_t\} \to \varphi_t \text{ weakly star in } L^\infty(0, T, L^2(0, L)),
\]

\[
\{\varphi^n_{tt}\} \to \varphi_{tt} \text{ weakly star in } L^\infty(0, T, L^2(0, L)),
\]

\[
\{\psi^n\} \to \psi \text{ weakly star in } L^\infty(0, T, H^1_+(0, L)),
\]

\[
\{\psi^n_t\} \to \psi_t \text{ weakly star in } L^\infty(0, T, L^2(0, L)),
\]

\[
\{z^n\} \to z \text{ weakly star in } L^\infty(0, T, L^2((0, L) \times (0, 1))).
\]

Moreover, from (22) we have

\[
\{\varphi^n\} \text{ is bounded in } L^2(0, T, H^1_0(0, L)),
\]

\[
\{\varphi^n_t\} \text{ is bounded in } L^2(0, T, L^2(0, L)).
\]

And since \(H^1_0(0, L)\) is compactly injected into \(L^2(0, L)\), see [6], we have by the Aubin-Lions theorem [7] that

\[
\{\varphi^n\} \to \varphi \text{ strongly in } L^\infty(0, T, L^2(0, L)).
\]

We also show that

\[
\{\varphi^n_t\} \to \varphi_t \text{ strongly in } L^\infty(0, T, L^2(0, L)),
\]

\[
\{\psi^n\} \to \psi \text{ strongly in } L^\infty(0, T, H^1_+(0, L)).
\]

Then we can pass to limit the approximate problem (18) – (19) in order to get a weak solution of problem (7).

And we use density arguments to get problems (7) that admit a global weak solution satisfying

\[
\begin{align*}
\varphi & \in L^\infty(0, T, H^1_0(0, L)), \\
\varphi_t & \in L^\infty(0, T, L^2(0, L)), \\
\psi & \in L^\infty(0, T, H^1_+(0, L)), \\
z & \in L^\infty(0, T, L^2((0, L) \times (0, 1))).
\end{align*}
\]  

(27)
Step 5. A Priori Estimate III

Suppose that the initial data in the approximate problem (18) satisfies
\((\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)\) (28)
strongly in \(H_1\).

Replacing \(u\) by \(-\varphi_{xxx}^n\) in (18)₁, \(v\) by \(-\psi_{xxx}^n\) in (18)₂ and \(w\) by \(z_{xx}^n\) in (18)₃
we arrive at
\[
\frac{d}{dt} F^n(t) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^L |z_{xx}^n(x, 0, t)|^2 dx
- \frac{\mu_1}{2} \int_0^L |z^n_{xx}(x, 1, t) + z^n_{xx}(x, 0, t)|^2 dx
+ \left[ \frac{\tau\mu_1 - \zeta}{2\tau} \right] \int_0^L |z_{xx}^n(x, 1, t)|^2 dx,
\]
(29)
where
\[
F^n(t) = \frac{\rho_1}{2} \int_0^L |\varphi_{tx}^n|^2 dx + \frac{b}{2} \int_0^L |\psi_{xx}^n|^2 dx + \frac{\kappa}{2} \int_0^L |\varphi_{tx}^n + \psi_{xx}^n|^2 dx
+ \frac{\zeta}{2} \int_0^L \int_0^1 |z_{tx}^n|^2 d\rho dx.
\]
Thus integrating (29) from 0 to \(t\), we obtain from our choice of initial data that
for all \(t \in [0; T]\) and for every \(n \in \mathbb{N}^*\),
\[
F^n(t) - F^n(0) = \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^t \int_0^L |z_{xx}^n(x, 0, s)|^2 dx ds
- \frac{\mu_1}{2} \int_0^t \int_0^L |z^n_{xx}(x, 1, s) + z^n_{xx}(x, 0, s)|^2 dx ds
- \frac{\mu_1}{2} \int_0^t \int_0^L |z^n_{xx}(x, 1, s) + z^n_{xx}(x, 0, s)|^2 dx ds.
\]
This means
\[
F^n(t) \to - \left[ \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \right] \int_0^t \int_0^L |z_{xx}^n(x, 0, s)|^2 dx ds
+ \frac{\mu_1}{2} \int_0^t \int_0^L |z^n_{xx}(x, 1, s) + z^n_{xx}(x, 0, s)|^2 dx ds.
\]
\[- \left[ \frac{\tau \mu_1 - \zeta}{2\tau} \right] \int_0^t \int_0^L |z^n_x(x,1,s)|^2 \, dx \, ds = F^n(0),\]

where

\[
F^n(0) = \frac{\rho_1}{2} \int_0^L |\varphi^n_{tx}(x,0)|^2 \, dx + \frac{b}{2} \int_0^L |\psi^n_{xx}(x,0)|^2 \, dx
+ \frac{\kappa}{2} \int_0^L |\varphi^n_{xx}(x,0) + \psi^n_x(x,0)|^2 \, dx
+ \frac{\zeta}{2} \int_0^L \int_0^1 |z^n_x(x,\rho,0)|^2 \, d\rho \, dx.
\]

As \((\varphi^n(0), \varphi^n_t(0), \psi^n(0), z^n(0)) = (\varphi^n_0, \varphi^n_1, \psi^n_0, z^n_0) \rightarrow (\varphi_0, \varphi_1, \psi_0, f_0)\) strongly in \(H_1\) we can deduce that each of the sequences \(\{\varphi^n(0)\}, \{\varphi^n_t(0)\}, \{\psi^n(0)\}\) and \(\{z^n(0)\}\) is bounded.

Thus there exists a positive constant \(C_3\) such that \(F^n(0) \leq C_3\). Hence

\[
F^n(t) = \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \int_0^t \int_0^L |z^n_x(x,0,s)|^2 \, dx \, ds
+ \frac{\mu_1}{2} \int_0^L \int_0^1 |z^n_x(x,1,s) + z^n_x(x,0,s)|^2 \, dx \, ds
- \left[ \frac{\tau \mu_1 - \zeta}{2\tau} \right] \int_0^t \int_0^L |z^n_x(x,1,s)|^2 \, dx \, ds \leq C_3.
\]

Which means that

\[
\frac{\rho_1}{2} \int_0^L |\varphi^n_{tx}|^2 \, dx + \frac{b}{2} \int_0^L |\psi^n_{xx}|^2 \, dx + \frac{\kappa}{2} \int_0^L |\varphi^n_{xx} + \psi^n_x|^2 \, dx
+ \frac{\mu_1}{2} \int_0^L \int_0^1 |z^n_x(x,1,s) + z^n_x(x,0,s)|^2 \, dx \, ds
- \left[ \frac{\tau \mu_1 - \zeta}{2\tau} \right] \int_0^t \int_0^L |z^n_x(x,1,s)|^2 \, dx \, ds \leq C_3,
\]

where \(C_3\) is a positive constant independent of \(t\) and \(n\) but depending on initial data. Then we can conclude that
\[\{\varphi^n\} \text{ is bounded in } L^\infty(0, T, H^2(0, L) \cap H_0^1(0, L)), \]
\[\{\varphi^n\} \text{ is bounded in } L^\infty(0, T, H_0^1(0, L)), \]
\[\{\psi^n\} \text{ is bounded in } L^\infty(0, T, H_*^2(0, L)), \]
\[\{z^n\} \text{ is bounded in } L^\infty(0, T, H^1((0, L) \times (0, 1))). \]

This implies that
\[\{\varphi^n\} \to \varphi \text{ weakly star in } L^\infty(0, T, H^2(0, L) \cap H_0^1(0, L)), \]
\[\{\varphi^n\} \to \varphi_t \text{ weakly star in } L^\infty(0, T, H_0^1(0, L)), \]
\[\{\psi^n\} \to \psi \text{ weakly star in } L^\infty(0, T, H_*^2(0, L)), \]
\[\{z^n\} \to z \text{ weakly star in } L^\infty(0, T, H^1((0, L) \times (0, 1))). \]

From the above limits, we conclude that \((\varphi, \varphi_t, \psi, z)\) is a strong weak solution satisfying
\[
\begin{align*}
\varphi &\in L^\infty(0, T, H^2(0, L) \cap H_0^1(0, L)), \\
\varphi_t &\in L^\infty(0, T, H_0^1(0, L)), \\
\psi &\in L^\infty(0, T, H_*^2(0, L)), \\
z &\in L^\infty(0, T, H^1((0, L) \times (0, 1))).
\end{align*}
\]

(31)

**Step 6. Continuous dependence**

Let \(U(t) = (\varphi, \varphi_t, \psi, z)\) and \(V(t) = (\varphi', \varphi'_t, \psi', z')\) be the stronger weak solutions of the problem (7) corresponding to initial data

\[U(0) = (\varphi_0, \varphi_1, \psi_0, z_0), \quad V(0) = (\varphi'_0, \varphi'_1, \psi'_0, z'_0) \in \mathcal{H}_1.\]

Then \((\Phi, \Phi_t, \Psi, Z) = U(t) - V(t)\) is solution of the system
\[
\begin{cases}
\rho_1 \Phi_{tt}(x, t) - \kappa(\Phi_x(x, t) + \Psi(x, t))_x + \mu_0 \Phi_t(x, t) + \mu_1 Z(x, 1, t) = 0 & \text{in } ]0, L[ \times (0, +\infty), \\
-b \Phi_{xx}(x, t) + \kappa(\Phi_x(x, t) + \Psi(x, t)) = 0 & \text{in } ]0, L[ \times (0, +\infty), \\
\tau Z_t(x, \rho, t) + Z_\rho(x, \rho, t) = 0 & (x, \rho, t) \in (0, L) \times (0, 1) \times (0, +\infty),
\end{cases}
\]

(32)

with initial data \((\Phi(0), \Phi_t(0), \Psi(0), Z(0)) = U(0) - V(0).\)

Multiplying (32)\(_1\) by \(\Phi_t\) and (32)\(_2\) by \(\Psi_t\) and integrating, we obtain
\[
\frac{d}{dt} \mathcal{G}(t) + \mu_1 \int_0^L \Phi_t(x, t) Z(x, 1, t) dx + \mu_0 \int_0^L |\Phi_t(x, t)|^2 dx = 0
\]
with
\[
G(t) = \frac{\rho_1}{2} \int_0^L |\Phi_t(x,t)|^2 dx + \frac{k}{2} \int_0^L |\Phi_x(x,t) + \Psi(x,t)|^2 dx \\
+ \frac{b}{2} \int_0^L |\Psi_x(x,t)|^2 dx.
\]

Applying Young’s inequality, we obtain the existence of a constant \(M_1\) such that
\[
\frac{d}{dt} G(t) \leq M_1 \int_0^L |\Phi_t(x,t)|^2 dx \\
\leq M_1 \left[ \int_0^L |\Phi_t(x,t)|^2 dx + \int_0^L |\Phi_x(x,t) + \Psi(x,t)|^2 dx \\
+ \int_0^L |\Psi_x(x,t)|^2 dx \right].
\]
(33)

And by integrating (33) from 0 to \(t\) we get
\[
G(t) \leq G(0) + M_1 \int_0^t \left[ \int_0^L |\Phi_{\lambda}(x,\lambda)|^2 dx + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2 dx \\
+ \int_0^L |\Psi_x(x,\lambda)|^2 dx \right] d\lambda.
\]
(34)

On the other hand, we know that for \(M_2 = \min \left\{ \frac{\rho_1}{2}, \frac{k}{2}, \frac{b}{2} \right\} \), we have
\[
G(\lambda) \geq M_2 \left[ \int_0^L |\Phi_{\lambda}(x,\lambda)|^2 dx + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2 dx \\
+ \int_0^L |\Psi_x(x,\lambda)|^2 dx \right].
\]
(35)

From (34) and (35) we have
\[
G(t) \leq G(0) + \frac{M_1}{M_2} \int_0^t G(\lambda) d\lambda.
\]
(36)

Applying Gronwall’s inequality we have
\[
G(t) \leq G(0) e^{\frac{M_1}{M_2} t}.
\]
(37)

So we obtain the continuous dependence of solution on the initial data. In particular, the solution is unique.
The proof of theorem is thus completed.  

4. Exponential stability

Theorem 4. Let the assumption (5) be satisfied. Then, there exist positive constants M and K such that, for any solution of (7)

\[ E(t) \leq ME(0)e^{-Kt}, \quad \forall t \geq 0. \]

Proof. Let \( F(t) = 2c\rho_1 \int_0^L \varphi_t \varphi dx + \mu_0 c \int_0^L |\varphi|^2 dx, \)

where \( c \) is a constant whose conditions we will specify later.

Multiply (7) by \( 2c \varphi \) and integrate from 0 to \( L \). We obtain

\[
2c\rho_1 \int_0^L \varphi_{tt}(x,t)\varphi(x,t)dx - 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]_x \varphi(x,t)dx + 2c\mu_0 \int_0^L \varphi_t(x,t)\varphi(x,t)dx + 2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0.
\]

As \( \varphi_{tt}\varphi = \frac{\partial}{\partial t}(\varphi_t \varphi) - |\varphi_t|^2 \), we have

\[
2c\rho_1 \int_0^L \frac{\partial}{\partial t}[\varphi_t(x,t)\varphi(x,t)]dx - 2c\rho_1 \int_0^L |\varphi_t|^2 dx + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)]_x \varphi(x,t)dx + 2c\mu_0 \int_0^L \frac{\partial}{\partial t}|\varphi(x,t)|^2 dx + 2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0,
\]

which means that

\[
-2c\rho_1 \int_0^L |\varphi_t|^2 dx + 2c\kappa \int_0^L [\varphi_x(x,t) + \psi(x,t)] \varphi_x(x,t)dx + 2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0.
\]

(38)
Multiply (7) by \(2c\psi\) and integrate over 0 to \(L\). We obtain

\[
-2cb \int_0^L \psi_{xx}(x, t) \psi \, dx + 2ck \int_0^L [\varphi_x(x, t) + \psi(x, t)] \psi \, dx = 0.
\]

Integrating by parts, we have

\[
-2cb [\psi_x \psi]_0^L + 2cb \int_0^L |\psi_x|^2 \, dx + 2ck \int_0^L [\varphi_x(x, t) + \psi(x, t)] \psi \, dx = 0.
\]

Since \(\psi_x\) is zero at 0 and \(L\), we have

\[
2cb \int_0^L |\psi_x|^2 \, dx + 2ck \int_0^L [\varphi_x(x, t) + \psi(x, t)] \psi \, dx = 0. \quad (39)
\]

Now summing (38) and (39), we obtain that

\[
\frac{d}{dt} F(t) = 2c\rho_1 \int_0^L |\varphi_t|^2 \, dx - 2ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 \, dx
\]

\[
- 2c\mu_1 \int_0^L z(x, 1, t) \varphi(x, t) \, dx - 2cb \int_0^L |\psi_x|^2 \, dx.
\]

Applying Young’s inequality to \(-2c\mu_1 \int_0^L z(x, 1, t) \varphi(x, t) \, dx\), we get

\[
\frac{d}{dt} F(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 \, dx - 2ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 \, dx
\]

\[
+ \frac{(2c\mu_1)^2}{4\varepsilon} \int_0^L |z(x, 1, t)|^2 \, dx + \varepsilon \int_0^L |\varphi(x, t)|^2 \, dx
\]

\[
- 2cb \int_0^L |\psi_x|^2 \, dx, \quad \forall \varepsilon > 0.
\]

Applying Poincare’s inequality at \(\varepsilon \int_0^L |\varphi(x, t)|^2 \, dx\), we obtain

\[
\frac{d}{dt} F(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 \, dx - 2ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 \, dx
\]
\[ + \varepsilon L^2 \int_0^L |\varphi_x(x, t)|^2 dx + \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x, 1, t)|^2 dx \]
\[ - 2cb \int_0^L |\psi_x|^2 dx, \quad \forall \varepsilon > 0. \quad (40) \]

We also know that
\[
\int_0^L |\varphi_x(x, t)|^2 dx \leq 2 \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx + 2 \int_0^L |\psi(x, t)|^2 dx.
\]

By Poincare’s inequality we have
\[
\int_0^L |\varphi_x(x, t)|^2 dx \leq 2 \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx + 2L^2 \int_0^L |\psi_x(x, t)|^2 dx.
\]  

(41)

Multiplying (41) by \(\varepsilon L^2\) we have

\[
\varepsilon L^2 \int_0^L |\varphi_x(x, t)|^2 dx \leq 2\varepsilon L^2 \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx + 2\varepsilon L^4 \int_0^L |\psi_x(x, t)|^2 dx
\]
\[
+ \frac{2\varepsilon L^2}{k} k \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx
\]
\[
+ \frac{2\varepsilon L^4}{b} b \int_0^L |\psi_x(x, t)|^2 dx.
\]

If we put \(c = \max\left\{ \frac{2\varepsilon L^2}{k}, \frac{2\varepsilon L^4}{b} \right\}\) we obtain

\[
\varepsilon L^2 \int_0^L |\varphi_x(x, t)|^2 dx \leq ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx + cb \int_0^L |\psi_x(x, t)|^2 dx.
\]  

(42)

From (40) and (42) we can write

\[
\frac{d}{dt} F(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 dx - 2ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx
\]
+ \( ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx \)
+ \( cb \int_0^L |\psi_x(x, t)|^2 dx + \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x, 1, t)|^2 dx \)
− \( 2cb \int_0^L |\psi_x|^2 dx, \ \forall \varepsilon > 0, \)

which means that

\[
\frac{d}{dt} F(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 dx - ck \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx \\
+ \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x, 1, t)|^2 dx \\
- cb \int_0^L |\psi_x|^2 dx, \ \forall \varepsilon > 0. \tag{43}
\]

Let us put \( I(t) = c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau \rho} |z|^2 d\rho dx, \) we have

\[
\frac{d}{dt} I(t) = 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau \rho} zz_t d\rho dx.
\]

From (7) we have \( z_t = -\frac{1}{\tau} z_\rho, \) so

\[
\frac{d}{dt} I(t) = \frac{-2c\zeta e^{2\tau}}{\tau} \int_0^L \int_0^1 e^{-2\tau \rho} zz_\rho d\rho dx \\
- 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau \rho} z^2 d\rho dx \\
- \frac{c\zeta e^{2\tau}}{\tau} \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau \rho} z^2\right) d\rho dx \\
+ \frac{c\zeta e^{2\tau}}{\tau} \int_0^L e^0 z^2(x, 0, t) dx \\
= \frac{-2c\zeta e^{2\tau}}{\tau} \int_0^L \int_0^1 e^{-2\tau \rho} z^2 d\rho dx - \frac{c\zeta e^{2\tau}}{\tau} \int_0^L e^{-2\tau} z^2(x, 1, t) dx \\
+ \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x, 0, t) dx \\
+ \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x, 0, t) dx
\]
\[
\begin{align*}
&\leq -\ c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau \rho} \rho z^2 d\rho dx - \frac{c\zeta}{\tau} \int_0^L z^2(x, 1, t) dx \\
&\quad + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x, 0, t) dx \\
&\quad - \ c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau \rho} \rho z^2 d\rho dx - \frac{c\zeta}{\tau} \int_0^L z^2(x, 1, t) dx \\
&\quad + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x, 0, t) dx,
\end{align*}
\]

however

\[
\frac{d}{dt} I(t) \leq -c\zeta \int_0^L \int_0^1 \rho z^2 d\rho dx - \frac{c\zeta}{\tau} \int_0^L z^2(x, 1, t) dx \\
\quad + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x, 0, t) dx.
\]

Summing (43) and (44) we have

\[
\begin{align*}
\frac{d}{dt} \left( I(t) + F(t) \right) &\leq 2c\rho_1 \int_0^L |\varphi_t|^2 dx - c\kappa \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx \\
&\quad + \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x, 1, t)|^2 dx - cb \int_0^L |\psi_x|^2 dx \\
&\quad - c\zeta \int_0^L \int_0^1 |z|^2 d\rho dx - \frac{c\zeta}{\tau} \int_0^L |z(x, 1, t)|^2 dx \\
&\quad + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L |z(x, 0, t)|^2 dx \\
&\quad - c \left[ \rho_1 \int_0^L |\varphi_t|^2 dx + b \int_0^L |\psi|^2 dx + \zeta \int_0^L \int_0^1 |z|^2 d\rho dx \\
&\quad + \kappa \int_0^L |\varphi_x(x, t) + \psi(x, t)|^2 dx \right] \\
&\quad + 3c\rho_1 \int_0^L |\varphi_t|^2 dx + \left[ \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} \right] \int_0^L |z(x, 1, t)|^2 dx \\
&\quad + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L |z(x, 0, t)|^2 dx - 2cE(t)
\end{align*}
\]
\[
\leq -2cE(t) + 3c\rho_1 \int_0^L |z(x,0,t)|^2 \, dx \\
+ \left[ \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} \right] \int_0^L |z(x,1,t)|^2 \, dx \\
+ \frac{c\zeta e^{2\tau}}{\tau} \int_0^L |z(x,0,t)|^2 \, dx.
\]

Thereby,
\[
\frac{d}{dt} \left( I(t) + F(t) \right) \leq -2cE(t) + \left( \frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1 \right) \int_0^L |z(x,0,t)|^2 \, dx \\
+ \left[ \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} \right] \int_0^L |z(x,1,t)|^2 \, dx.
\]

(45)

Setting
\[
L(t) = NE(t) + I(t) + F(t), \quad \forall t > 0,
\]
where \( N \) is a constant whose conditions we will specify later. We obtain from (14) and (45)
\[
\frac{d}{dt} L(t) \leq \left[ N \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} + \frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1 \right] \int_0^L |z(x,0,t)|^2 \, dx \\
- \frac{N\mu_1}{2} \int_0^L |z(x,1,t) - z(x,0,t)|^2 \, dx - 2cE(t) \\
+ \left[ N \frac{\tau\mu_1 - \zeta}{2\tau} + \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} \right] \int_0^L |z(x,1,t)|^2 \, dx.
\]

As \( \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \leq 0; -\frac{\mu_1}{2} \leq 0 \) and \( \frac{\tau\mu_1 - \zeta}{2\tau} \leq 0 \), just take \( N \) very large enough such that
\[
N \frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} + \frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1 < 0,
\]
and
\[
N \frac{\tau\mu_1 - \zeta}{2\tau} + \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} < 0.
\]

Now we can take
\[
N > \max \left\{ \frac{6c\tau\rho_1 + 2c\zeta e^{2\tau}}{\tau(2\mu_0 - \mu_1) - \zeta}, \frac{2\tau c^2 \mu_1^2 - 2c\zeta \varepsilon}{(\zeta - \tau\mu_1)\varepsilon} \right\}.
\]
For this value of $N$ we obtain that

$$\frac{d}{dt} \mathcal{L}(t) \leq -2cE(t).$$

(46)

Moreover, it is easy to see the existence of two constants $\alpha$ and $\beta$ such that

$$\alpha E(t) \leq \mathcal{L}(t) \leq \beta E(t), \forall t \geq 0.$$ 

(47)

By (46) and (47) we obtain

$$\frac{d}{dt} \frac{\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2cE(t)}{\mathcal{L}(t)} \leq \frac{-2cE(t)}{\beta E(t)},$$

which means that

$$\frac{d}{dt} \frac{\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2c}{\beta}.$$ 

(48)

Integrating (48) from 0 to $t$ we have

$$\ln \mathcal{L}(t) - \ln \mathcal{L}(0) \leq \frac{-2c}{\beta} t.$$ 

(49)

By (47) we obtain

$$\alpha E(t) \leq \mathcal{L}(t),$$

and

$$\mathcal{L}(0) \leq \beta E(0).$$

However (49) can be written as

$$E(t) \leq \frac{\beta E(0)}{\alpha} e^{-\frac{2c}{\beta} t}.$$ 

We obtain, by taking $M = \frac{\beta}{\alpha}$ and $K = \frac{2c}{\beta}$, that

$$E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0.$$ 

The proof of theorem is thus completed.
References


