International Journal of Applied Mathematics
Volume 25 No. 1 2012, 109-124

FORWARD (α, β)-DIFFERENCE OPERATOR ND ITS SOME APPLICATIONS IN NUMBER THEORY

G. Britto Antony Xavier[1 §], P. Rajiniganth[2],
M. Maria Susai Manuel[3], V. Chandrasekar[4]

[1,4] Department of Mathematics
Sacred Heart College
Tirupattur - 635601, Vellore District
Tamil Nadu, S. INDIA
[1]e-mail: shcbritto@yahoo.co.in

[2]Department of Science and Humanities
Saveetha School of Engineering
Saveetha University
Chennai-602 105, Tamil Nadu, S. INDIA

[3]Department of Science and Humanities
R.M.D. Engineering College
Kavaraipettai - 601 206, Tamil Nadu, S. INDIA

Abstract: In this paper, we extend the theory of the forward difference operator Δ_h to the forward (α, β)-difference operator $\Delta_{(\alpha,\beta)(h)}$, and present the discrete version of the Leibnitz Theorem and Binomial Theorem with reference to $\Delta_{(\alpha,\beta)(h)}$. By defining its inverse, the formulas for sum of various types of higher powers of arithmetic-double geometric progressions in number theory are established as applications of $\Delta_{(\alpha,\beta)(h)}$.

AMS Subject Classification: 39A12, 39A70, 47B39, 39B60
Key Words: forward (α, β)-difference operator, arithmetic-double geometric series

Received: December 29, 2011

[§]Correspondence author

1. Introduction

The difference equation describes the evolution of a certain phenomena over a period of time. The theory of difference equations is based on the operator Δ defined as

$$\Delta f(t) = f(t+1) - f(t), \qquad t \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}.$$

The forward h-difference operator Δ_h defined as

$$h\Delta_h f(t) = f(t+h) - f(t), \quad t \in [0, \infty),\ h \in (0, \infty)$$

can be found in Difference Calculus, in particular in Fracional Difference Calculus, see [1, 4, 5].

In [9], Miller and Ross defined a fractional sum of order $\nu > 0$ via the solution of a linear difference equation. They introduce it as

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s). \tag{1}$$

Equation (1) is discrete analogue to the Riemann-Liouville fractional integral

$${}_aD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (t - \sigma(s))^{\nu-1} f(s) ds$$

of order $\nu > 0$, which can be obtained via the solution of a linear differential equation [9, 10]. The basic properties of the operator $\Delta^{-\nu}$ (1) have been obtained in [9]. More recently, Atici and Eloe introduced the fractional difference of order $\mu > 0$ by $\Delta^{\mu} f(t) = \Delta^m(\Delta^{\mu-m} f(t))$, where m is the integer part of μ, and developed some of its properties that allow to obtain solutions of certain fractional difference equations [2, 3]. Since equation (1) yields more results in difference equations rather than in number theory, we make use of the forward (α, β)-difference operator and its inverse, similar to (1), to obtain significant formulas on certain types of finite and infinite series in number theory.

Throughout this paper, we assume the following notations:

(i) $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $\mathbb{N}_h(j) = \{j, j+h, j+2h, ...\}$,

(ii) α, β, h are positive reals and m, n, r are positive integers, and

(iii) $[x]$ is the integer part of x and $rC_i = \dfrac{r!}{(r-i)!i!}$.

2. Relations Among Difference Operators and Shift Operator

In this section, we present some basic definitions and preliminary results which will be useful in the further discussion.

Definition 2.1. Let $f(t)$ be real or complex valued function on $[0, \infty)$. Then, the forward (α, β)-difference operator $\Delta_{(\alpha,\beta)(h)}$ on $f(t)$ is defined as

$$h\Delta_{(\alpha,\beta)(h)} f(t) = \beta f(t+h) - \alpha f(t), \quad h \in (0, \infty). \tag{2}$$

Remark 2.2. i) When $\beta = 1$, the difference operator $h\Delta_{(\alpha,1)(h)}$ becomes the generalized α-difference operator $h\Delta_{\alpha(h)}$ as $h\Delta_{\alpha(h)} f(t) = f(t+h) - \alpha f(t)$, see [8].

ii) When $\alpha = 1$, and $\beta = 1$ the difference operator $\Delta_{(1,1)(h)}$ becomes the generalized difference operator $h\Delta_h$ as $h\Delta_h f(t) = f(t+h) - f(t)$, see [6].

iii) When $\alpha = 1$, $\beta = 1$ and $h = 1$, $\Delta_{(1,1)(1)}$ is nothing but the usual difference operator Δ [1].

iv) The usual shift operator E satisfies $E^h f(t) = f(t+h)$.

The following are simple deductions using the definition of $\Delta_{(\alpha,\beta)(h)}$:
The operators $\Delta_{(\alpha,\beta)(h)}$, Δ_h and E satisfy the following relations:

$$(i) \quad E^h = \beta^{-1}[h\Delta_{(\alpha,\beta)(h)} + \alpha] = (1 + h\Delta)^h = 1 + h\Delta_h, \tag{3}$$

$$(ii) \quad \alpha + m\Delta_{(\alpha,\beta)(m)} = \beta \sum_{i=0}^{m} mC_i \Delta^i, \frac{1}{\beta}\Delta_{(\alpha,\beta)(m)} = \Delta_{\frac{\alpha}{\beta}(m)},$$

$$(iii) \quad \Delta_{(\alpha,\beta)(h)}(c_1 f(t) + c_2 g(t)) = c_1 \Delta_{(\alpha,\beta)(h)} f(t) + c_2 \Delta_{(\alpha,\beta)(h)} g(t), \tag{4}$$

$$(iv) \quad (h\Delta_{(\alpha,\beta)(h)})^n = \sum_{r=0}^{n} nC_r (-\alpha)^r (\beta)^{n-r} E^{h(n-r)}, \tag{5}$$

and hence

$$(h\Delta_{(\alpha,\beta)(h)})^n f(t) = \sum_{r=0}^{n} nC_r (-\alpha)^r (\beta)^{n-r} f(t + h(n-r)), \tag{6}$$

(*v*) $\alpha + (h_1 + h_2 + \ldots + h_n)\Delta_{(\alpha,\beta)(h_1+h_2+\ldots+h_n)} = \prod_{i=1}^{n}(\alpha + h_i\Delta_{(\alpha,\beta)(h_i)})$, h_i's are real, and

$$(vi) \quad h\Delta_{(\alpha,\beta)(h)} = \beta(1 + h\Delta)^h - \alpha,\ h\Delta_{(\alpha,\beta)(nh)} = \beta(1 + h\Delta_h)^n - \alpha. \tag{7}$$

The discrete version of the Leibnitz theorem according to $\Delta_{(\alpha,\beta)(h)}$ is given below.

Theorem 2.3. *If $f(t)$ and $g(t)$, $t \in [0, \infty)$ are two real or complex valued functions, then*

$$\Delta^n_{(\alpha,\beta)(h)}(f(t)g(t)) = \sum_{r=0}^{n} nC_r\alpha^{n-r}\Delta^r_{(\alpha,\beta)(h)}f(t)\Delta_h^{n-r}g(t + rh).$$

Proof. The operators E_1^h, E_2^h defined as

$$E_1^h(f(t)g(t)) = f(t + h)g(t), \quad E_2^h(f(t)g(t)) = f(t)g(t + h) \tag{8}$$

yield

$$E^h = E_1^h E_2^h. \tag{9}$$

Again by defining

$$(h\Delta_{(\alpha,\beta)(h)})_1 = \beta E_1^h - \alpha \quad \text{and} \quad (\mathrm{h}\Delta_{(\alpha,\beta)(\mathrm{h})})_2 = \beta \mathrm{E}_2^{\mathrm{h}} - \alpha \tag{10}$$

for the functions $f(t)$ and $g(t)$ respectively, we obtain

$$h\Delta_{(\alpha,\beta)(h)} = \beta E^h - \alpha = \beta E_1^h E_2^h - \alpha.$$

From (3), we get

$$h\Delta_{(\alpha,\beta)(h)} = [(h\Delta_{(\alpha,\beta)(h)})_1 + \alpha]E_2^h - \alpha,$$

$$h\Delta_{(\alpha,\beta)(h)} = (h\Delta_{(\alpha,\beta)(h)})_1 E_2^h + \alpha(h\Delta_h)_2,$$

where $E_2^h - 1 = (h\Delta_h)_2$ (see Theorem 2.5, [6]).

Hence, we find

$$(h\Delta_{(\alpha,\beta)(h)})^n(f(t)g(t)) = \{\alpha(h\Delta_h)_2 + (h\Delta_{(\alpha,(h)})_1 E_2^h\}^n(f(t)g(t)). \tag{11}$$

The proof follows by using the Binomial theorem, (8), (9), (10) and (11). □

Lemma 2.4. *If $h \in (0,\infty)$ and n is a positive integer, then*

$$E^{nh} = \beta^{-n}\sum_{r=0}^{n} nC_r\alpha^{n-r}(h\Delta_{(\alpha,\beta)(h)})^r. \tag{12}$$

Proof. The proof folows from (3). □

The following is the discrete version of the generalized Binomial theorem involving $\Delta_{(\alpha,\beta)(h)}$.

Theorem 2.5. *If m and n are positive integers, then*

$$\beta^n(t+nh)^m = \sum_{r=0}^{n} nC_r\alpha^{n-r}(h\Delta_{(\alpha,\beta)(h)})^r t^m. \tag{13}$$

Proof. The proof follows by operating (6) and (12) on $f(t) = t^m$. □

Example 2.6. If θ is in degrees assuming only integer values in the anticlockwise direction, then

$$\beta^n \sin(t+n\theta) = \sum_{r=0}^{n} nC_r\alpha^{n-r}(h\Delta_{(\alpha,\beta)(\theta)})^r sin(t+r\theta).$$

Proof. The proof follows by operating (6) and (12) on $f(t) = \sin t$ and taking $h = \theta$. □

The following theorem establishes a generalized version of Montmorte's theorem with reference to $\Delta_{(\alpha,\beta)(h)}$.

Theorem 2.7. *If the series $\sum\limits_{t=0}^{\infty} f(th)x^{th}$ converges, then it can be expressed as*

$$\sum_{t=0}^{\infty} f(th)x^{th} = \beta\sum_{t=0}^{\infty}\frac{x^{th}(h\Delta_{(\alpha,\beta)(h)})^t f(0)}{(\beta-\alpha x^h)^{t+1}}. \tag{14}$$

Proof. Using the definition of the shift operator E, we get

$$\sum_{t=0}^{\infty} f(th)x^{th} = \sum_{t=0}^{\infty} x^{th}E^{th}f(0) = \{1 - x^hE^h\}^{-1}f(0).$$

Now, the proof follows from the relation $E^h = \beta^{-1}[\alpha + h\Delta_{(\alpha,\beta)(h)}]$. □

Lemma 2.8. *Let* $t_h^{(n)} = t(t-h)\cdots(t-(n-1)h)$ *be a generalized polynomial factorial. Then,*

$$h\Delta_{(\alpha,\beta)(h)} t_h^{(n)} = [(\beta-\alpha)t + (\beta+\alpha(n-1))h] t_h^{(n-1)}. \tag{15}$$

Proof. The proof follows by (2) and the relation

$$t_h^{(n-1)} = t(t-h)(t-2h)\cdots(t+2h-nh).$$

□

Lemma 2.9. (see [6]) *For* $n \in N(1)$, *if* s_r^n *and* S_r^n *are the Stirling numbers of first and second kinds respectively, then*

$$t_h^{(n)} = \sum_{r=1}^{n} s_r^n h^{n-r} t^r; \quad t^n = \sum_{r=1}^{n} S_r^n h^{n-r} t_h^{(r)}. \tag{16}$$

Lemma 2.10. *Let* $f(t), t \in [0,\infty)$ *be a real or complex valued function and* h, α *are positive reals. Then*

$$\sum_{j=0}^{\infty} \frac{x^{jh} f(jh)}{j!h^j} = \left\{ e^{\frac{x^h \alpha}{\beta h}} e^{\frac{x^h h \Delta_{(\alpha,\beta)(h)}}{\beta h}} \right\} f(0). \tag{17}$$

Proof. From the shift operator, $E^{jh} f(0) = f(jh)$, we find that

$$\left[e^{\frac{x^h E^h}{h}}\right] f(0) = f(0) + \frac{x^h E^h}{1!h} f(0) + \frac{x^{2h} E^{2h}}{2!h^2} f(0) + \cdots = \sum_{j=0}^{\infty} \frac{x^{jh}}{j!h^j} f(jh).$$

Now, the proof follows from (3). □

3. Inverse of the Operator $\Delta_{(\alpha,\beta)(h)}$

In this section, we introduce the definition of inverse of the operator $\Delta_{(\alpha,\beta)(h)}$ and develop results involving $\Delta^{-1}_{\alpha,\beta)(h)}$.

Definition 3.1. The inverse of the operator $\Delta_{(\alpha,\beta)(h)}$ denoted by $\Delta^{-1}_{(\alpha,\beta)(h)}$ is defined as follows. If $h\Delta_{(\alpha,\beta)(h)}v(t) = f(t)$ and $j = t - \left[\frac{t}{h}\right]h$, then

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}f(t)\Big|_j^t = v(t) - \left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v(j), \tag{18}$$

where $v(j)$ is constant for all $t \in N_h(j), \beta \neq 0$ and the n^{th} order inverse operator denoted by $\Delta^{-n}_{(\alpha,\beta)(h)}$ is defined as,

$$\Delta^{-n}_{(\alpha,\beta)(h)}f(t) = \Delta^{-1}_{(\alpha,\beta)(h)}\Delta^{-(n-1)}_{(\alpha,\beta)(h)}f(t),\ n \geq 2.$$

Example 3.2. Since $h\Delta_{(\alpha,\beta)(h)}\frac{t}{(\beta-\alpha)} = t + \frac{\beta h}{(\beta-\alpha)}$, $\beta \neq \alpha$ by Definition 3.1, we obtain

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}\left(t + \frac{\beta h}{\beta - \alpha}\right) = \frac{t}{(\beta-\alpha)} - \left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}\frac{j}{(\beta-\alpha)},$$

where $j = t - \left[\frac{t}{h}\right]h$.

Remark 3.3. $\Delta_{(\alpha,\beta)(h)}\Delta^{-1}_{(\alpha,\beta)(h)}f(t) \neq \Delta^{-1}_{(\alpha,\beta)(h)}\Delta_{(\alpha,\beta)(h)}f(t)$.

Lemma 3.4. *If α, β and h are positive reals and $t \in [h,\infty)$, then*

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}f(t)\Big|_j^t = \sum_{r=1}^{\left[\frac{t}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}f(t-rh). \tag{19}$$

Proof. The proof follows from the relations

$$h\Delta_{(\alpha,\beta)(h)}\left\{\sum_{r=1}^{\left[\frac{t}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}f(t-rh)\right\} = f(t),$$

$v(j) = \sum_{r=1}^{\left[\frac{j}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}f(j-rh) = 0$ as $j = t - \left[\frac{t}{h}\right]h < h$ and Definition 3.1. □

Lemma 3.5. *Let $\lambda \neq 1, t \geq 2\alpha h$ and $P(t)$ is a function in t. Then*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}\lambda^{t-rh}P(t-rh) = v(t) - \left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v(j),$$

where $v(t) = \dfrac{\lambda^t}{\alpha(\lambda^h - 1)} \sum_{t=0}^{\infty} (-1)^t \Big[\dfrac{\lambda^h h\Delta_{(\alpha,\beta)(h)}}{\alpha(\lambda^h - 1)}\Big]^t P(t).$

Proof. For a function $F(t)$, we find

$$\begin{aligned} h\Delta_{(\alpha,\beta)(h)}(\lambda^t F(t)) &= \beta\lambda^{t+h} F(t+h) - \alpha\lambda^t F(t) \\ &= \lambda^t(\beta\lambda^h E^h - \alpha)F(t) = \lambda^t P(t), \end{aligned}$$

hence, we take

$$(\beta\lambda^h E^h - \alpha)^{-1} P(t) = F(t). \tag{20}$$

Since $h\Delta_{(\alpha,\beta)(h)}(\lambda^t F(t)) = \lambda^t P(t)$, by Definition 3.1, we get

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}(\lambda^t P(t)) = \lambda^t F(t) - \Big(\frac{\alpha}{\beta}\Big)^{\left[\frac{t}{h}\right]} \lambda^j F(j),$$

and hence from (20), we get

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}(\lambda^t P(t)) = \lambda^t(\beta\lambda^h E^h - \alpha)^{-1} P(t) - \Big(\frac{\alpha}{\beta}\Big)^{\left[\frac{t}{h}\right]} \lambda^j P(j), \tag{21}$$

where $j = t - \left[\frac{t}{h}\right]h$. The proof now follows from (3), (19), (20), (21) and the Binomial theorem. □

4. Applications of $\Delta^{-1}_{(\alpha,\beta)(h)}$ on finite terms of A.P.

In this section, we establish the formula for the sum of arithmetic - geometric progression as well as arithmetic - double geometirc progression in number theory, as an application of $\Delta_{(\alpha,\beta)(h)}$.

The following theorem is the general formula for the sum of the higher powers of arithmetic - geometric progression.

Theorem 4.1. *Let $t \in [h, \infty)$ and n is a positive integer and $\beta \neq 0$. Then,*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]} \frac{\alpha^{r-1}}{\beta^r}(t - rh)^n = v^n(t) - \Big(\frac{\alpha}{\beta}\Big)^{\left[\frac{t}{h}\right]} v^n(j), \tag{22}$$

where $v^n(t) = \dfrac{1}{\beta - \alpha}[t^n - n\beta h v^{n-1}(t) - \cdots - -\beta h^n v^0(t)]$,
$v^0(t) = v^0(j) = \frac{1}{\beta-\alpha}$ *and* $v^1(t), v^2(t), \cdots, v^{n-1}(t)$ *are obtained from the above expression by putting the corresponding values for* n.

Proof. The proof follows by taking $f(t)=t^n$ in Lemma 3.4 and (18). □

The following corollary illustrates Theorem 4.1 and gives formula for the sum of the cubes of an arithmetic - geometric progression.

Corollary 4.2. *Let $t\in[h,\infty)$, $\alpha\neq\beta$, $n=3$ and $j=t-\left[\frac{t}{h}\right]h$. Then*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}(t-rh)^3=v^3(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v^3(j). \tag{23}$$

Proof. Since $h\Delta_{(\alpha,\beta)(h)}(t^0)=\beta-\alpha$, Definition 3.1 yields

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}(t^0=1)=v^0(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v^0(j), \tag{24}$$

where $v^0(t)=v^0(j)=\frac{1}{\beta-\alpha}$. Since $h\Delta_{(\alpha,\beta)(h)}t=(\beta-\alpha)t+\beta h$ and $h\Delta_{(\alpha,\beta)(h)}$ is linear, Definition 3.1 and (24) give

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}t=v^1(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v^1(j), \tag{25}$$

where $v^1(t)=\frac{1}{\beta-\alpha}[t-\beta hv^0(t)]$. Since $h\Delta_{(\alpha,\beta)(h)}t^2=(\beta-\alpha)t^2+2t\beta h+\beta h^2$. Definition 3.1, (24) and (25) yield

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}t^2=v^2(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v^2(j), \tag{26}$$

where $v^2(t)=\frac{1}{\beta-\alpha}[t^2-2\beta hv^1(t)-\beta h^2v^0(t)]$. Similarly, since

$$h\Delta_{(\alpha,\beta)(h)}t^3=(\beta-\alpha)t^3+3\beta ht^2+3\beta h^2t+\beta h^3, \tag{27}$$

from Definition 3.1 and (27), we find

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}t^3=v^3(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}v^3(j), \tag{28}$$

where $v^3(t)=\frac{1}{\beta-\alpha}[t^3-3\beta hv^2(t)-3\beta h^2v^1(t)-\beta h^3v^0(t)]$.

Now, the proof follows by applying (19) for $(h\Delta_{(\alpha,\beta)(h)})^{-1}t^3$. □

The following example is an illustration of Corollary 4.2.

Example 4.3. The value of the cubes of an arithmetic-geometric series

$$A^3 = \frac{2^0}{3^1}(t-1(3))^3 + \frac{2^1}{3^2}(t-2(3))^3 + \cdots + \frac{2^{\left[\frac{t}{3}\right]-1}}{3^{\left[\frac{t}{3}\right]}}(t-[\frac{t}{3}]3)^3$$

is $v^3(t) - \left(\frac{2}{3}\right)^{\left[\frac{t}{3}\right]} v^3(j)$, where $h=3$, $\alpha=2$, $\beta=3$ and

$$v^3(t) = t^3 - 3(3)(3)v^2(t) - 3(3)(3)^2 v^1(t) - (3)(3)^3 v^0(t),$$
$$v^2(t) = t^2 - 2(3)(3)v^1(t) - 3(3)^2 v^0(t),$$
$$v^1(t) = t - 3(3)v^0(t) \quad \text{and} \quad v^0(t) = 1.$$

In particular, when $t = 46$, we find $A^3 = 55842.97862$.

Remark 4.4. Similarly, one can find A^n, for $n \in \mathbb{N}(0)$.

The following theorem gives formula for sum of products of terms of G.P. with m consecutive terms of A.P.

Theorem 4.5. *Let $t \in [h, \infty)$, $j = t - \left[\frac{t}{h}\right]h$ and n is a positive integer and $\beta \neq 0$. Then,*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]} \frac{\alpha^{r-1}}{\beta^r}(t-rh)_h^{(m)} = v^{(m)}(t) - \left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]} v^{(m)}(j), \tag{29}$$

where $v^{(m)}(t) = \sum_{i=1}^{m} \frac{(-1)^i m^{(i)} t_h^{(m-i)} (\beta h)^i}{(\beta-\alpha)^{i+1}}$, $\beta \neq \alpha$.

Proof. The proof follows by using the Stirling numbers of first kind, Lemma 3.4 and Theorem 4.1. □

Corollary 4.6. *Let $t \in [h, \infty)$, $j = t - \left[\frac{t}{h}\right]h$ and $m = 3$. Then*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]} \frac{\alpha^{r-1}}{\beta^r}(t-rh)_h^{(3)} = v^{(3)}(t) - \left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]} v^{(3)}(j), \tag{30}$$

where $v^{(3)}(t) = \frac{t_h^{(3)}}{(\beta-\alpha)} - \frac{3(\beta h)t_h^{(2)}}{(\beta-\alpha)^2} + \frac{3^{(2)}(\beta h)^2 t_h^{(1)}}{(\beta-\alpha)^3} - \frac{3^{(3)}(\beta h)^3 t_h^{(0)}}{(\beta-\alpha)^4}$.

The following example is an illustration of Corollary 4.6.

Example 4.7. By taking $h=2$, $\alpha=2$, $\beta=3$ and $t=201$ in (30), we obtain

$$\sum_{r=1}^{100}\frac{2^{r-1}}{3^r}(201-2r)^{(3)}=v^{(3)}(201)-(\frac{2}{3})^{100}v^{(3)}(1),$$

where $v^{(3)}(201)=(201)^{(3)}-(3)(6)(201)^{(2)}+(3)^{(2)}(6)^2(201)^{(1)}-(3)^{(3)}(6)^3$.

The following theorem is the formula for the sum of the arithmetic - double geometric progression.

Theorem 4.8. *If $\alpha\neq\beta a^h$, $t\in[h,\infty)$, $j=t-\left[\frac{t}{h}\right]h$ and $\beta\neq 0$, then*

$$\sum_{r=1}^{\left[\frac{t}{h}\right]}\frac{\alpha^{r-1}}{\beta^r}(t-rh)a^{t-rh}=w(t)-\left(\frac{\alpha}{\beta}\right)^{\left[\frac{t}{h}\right]}w(j), \tag{31}$$

where $w(t)=\dfrac{(t)a^t}{(\beta a^h-\alpha)}-\dfrac{\beta h a^{t+h}}{(\beta a^h-\alpha)^2}$.

Proof. Since

$$h\Delta_{(\alpha,\beta)(h)}\left(\frac{(t)a^t}{(\beta a^h-\alpha)}\right)=(t)a^t+\frac{\beta h a^{t+h}}{(\beta a^h-\alpha)},$$

Definition 3.1 yields

$$\frac{(t)a^t}{(\beta^h-\alpha)}-(\frac{\alpha}{\beta})^{\left[\frac{t}{h}\right]}\frac{\left(t-\left[\frac{t}{h}\right]h\right)a^{\left(t-\left[\frac{t}{h}\right]h\right)}}{(\beta a^h-\alpha)}$$

$$=(h\Delta_{(\alpha,\beta)(h)})^{-1}(ta^t)+(h\Delta_{(\alpha,\beta)(h)})^{-1}\left(\frac{\beta h a^{t+h}}{(\beta a^h-\alpha)}\right). \tag{32}$$

Also, since $h\Delta_{(\alpha,\beta)(h)}\left(\dfrac{a^k}{\beta a^h-\alpha}\right)=a^k$, by (18), we find

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}a^k=\frac{a^k}{(\beta a^h-\alpha)}-(\frac{\alpha}{\beta})^{\left[\frac{t}{h}\right]}\frac{a^{\left(k-\left[\frac{t}{h}\right]h\right)}}{(\beta a^h-\alpha)}. \tag{33}$$

Since $\Delta^{-1}_{(\alpha,\beta)(h)}$ is linear. The proof follows from (32), (33) and (19) for $f(t)=ta^t$. □

The following example illustrates Theorem 4.8.

Example 4.9. Consider the sum of the arithmetic - double geometric series $G_d = \frac{(5)^0}{3^1}(t-(1)(2))3^{t-(1)(2)} + \frac{(5)^1}{3^2}(t-(2)(2))3^{t-(2)(2)} + \frac{(5)^2}{3^3}(t-(3)(2))3^{t-(3)(2)} + \cdots + \frac{(5)^{\left[\frac{t}{2}\right]-1}}{(3)^{\left[\frac{t}{2}\right]}}(t-\left[\frac{t}{2}\right]2)(3)^{(t-\left[\frac{t}{2}\right]2)}$. Then $G_d = w(t) - \left(\frac{5}{3}\right)^{\left[\frac{t}{2}\right]} w(j)$, where $w(t) = \frac{(t)3^t}{22} - \frac{(3)(2)(3)^{t+2}}{(22)^2}$, by taking $h=2$, $a=3$, $\alpha=5$ and $\beta=3$ in (31).

In particular when $t=21$, we find

$$w(21) = \frac{(21)3^{21}}{22} - \frac{(6)3^{23}}{22^2}$$

,

$$w(1) = \frac{(1)(3)}{22} - \frac{(6)3^3}{22^2}$$

and

$$G_d = w(21) - \left(\frac{5}{3}\right)^{10} w(1) = 8817818435.$$

Similarly, one can use Theorem 4.5 and Theorem 4.8 to find several types of sum of products of terms of G.P. with consecutive terms of A.P. and Arithmetic - double Geometric progression respectively.

5. Applications of $\Delta^{-1}_{(\alpha,\beta)(h)}$ on infinite terms of A.P.

In this section, we establish the formula for the sum of an arithmetic - geometric progression as well as an arithmetic - double geometirc progression of infinite series in number theory as an application of $\Delta^{-1}_{(\alpha,\beta)(h)}$.

Lemma 5.1. *If* $\lim_{t\to\infty} f(t) = 0$, *then for* $\alpha > 1$, $\alpha > \beta$,

$$(h\Delta_{(\alpha,\beta)(h)})^{-1} f(t) = -\sum_{r=0}^{\infty} \frac{\beta^r}{\alpha^{r+1}} f(t+rh). \tag{34}$$

Proof. The proof follows from the relation

$$-h\Delta_{(\alpha,\beta)(h)} \sum_{r=0}^{\infty} \frac{\beta^r}{\alpha^{r+1}} f(t+rh) = f(t),$$

$v(j) = 0$ as $t \to \infty$ and Definition 3.1. □

Theorem 5.2. *If $\alpha > 1$ and $\alpha > \beta$, then*

$$\sum_{r=0}^{\infty} \frac{\beta^r}{\alpha^{r+1}} = \frac{1}{(\alpha-\beta)}. \tag{35}$$

Proof. The proof follows from the Definition 3.1, and by $f(t) = 1$ in Lemma 5.1. □

The following theorem gives the formula for sum of arithmetic geometric progression on infinite series.

Theorem 5.3. *If $\alpha > 1$, $\alpha > \beta$ and $t \in [0,\infty)$, then*

$$\sum_{r=0}^{\infty} \frac{\beta^r}{\alpha^{r+1}} f(t+rh) = \frac{1}{(\alpha-\beta)}\Big(t - \frac{\beta h}{\beta-\alpha}\Big). \tag{36}$$

Proof. Since $h\Delta_{(\alpha,\beta)(h)} t = (\beta-\alpha)\big(t + \dfrac{\beta h}{\beta-\alpha}\big)$ and $\Delta_{(\alpha,\beta)(h)}$ is linear, from Definition 3.1, we find

$$\frac{t}{(\beta-\alpha)} = (h\Delta_{(\alpha,\beta)(h)})^{-1} t + \frac{\beta h}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}(1),$$

which yields from (35),

$$(h\Delta_{(\alpha,\beta)(h)})^{-1} t = \frac{1}{(\beta-\alpha)}\left(t - \frac{\beta h}{\beta-\alpha}\right). \tag{37}$$

Now the proof follows by $f(t) = t$ in Lemma 5.1. □

The following example illustrates Theorem 5.3.

Example 5.4. Taking $h = 2$, $\alpha = 5$ and $\beta = 2$ in (36), we find $\sum\limits_{r=0}^{\infty} \frac{2^r}{5^{r+1}}(t+rh) = \frac{-1}{3}\Big(t + \frac{(2)(2)}{3}\Big)$. When $t = 2$, $\frac{2}{5} + \frac{2(4)}{5^2} + \frac{2^2(6)}{5^3} + \frac{2^3(8)}{5^4} + \cdots = \frac{-1}{3}\Big(2 + \frac{4}{3}\Big)$.

Theorem 5.5. *Let $t \in [0,\infty)$ and $h \in (0,\infty)$. Then,*

$$\sum_{r=0}^{\infty} \frac{\beta^r}{\alpha^{r+1}}(t+rh)^2 = \frac{t^2}{(\alpha-\beta)} + \frac{\beta h}{(\alpha-\beta)^2}\left[2t - \frac{2\beta h}{\beta-\alpha} + h\right]. \tag{38}$$

Proof. Since $\Delta_{(\alpha,\beta)(h)}$ is linear, from Definition 3.1,

$$\frac{t^2}{\beta-\alpha} = (h\Delta_{(\alpha,\beta)(h)})^{-1}t^2$$

$$+\frac{2\beta h}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}t + \frac{\beta h^2}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}(1). \tag{39}$$

By substituting (37), (35) in (39), we find

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}t^2 = \frac{t^2}{(\beta-\alpha)} - \frac{\beta h}{(\beta-\alpha)^2}\left[2t - \frac{2\beta h}{\beta-\alpha} + h\right]. \tag{40}$$

Now the proof follows by Lemma 5.1. □

The following is an illustration for Theorem 5.5.

Example 5.6. The sum of arithmetic-geometric progression for power 2,

$$\sum_{r=0}^{\infty}\frac{(1.5)^r}{(3.5)^{r+1}}(t+0.2r)^2 = \frac{t^2}{2} + \frac{(1.5)(0.2)}{2^2}\left[2t + \frac{2(1.5)(0.2)}{2} + 0.2\right]. \tag{41}$$

Solution. (41) follows by taking $h = 0.2$, $\alpha = 2.5$ and $\beta = 1.5$ in (38). In particular, when $t = 5$, we get

$$\frac{5^2}{3.5} + \frac{(1.5)(5.2)^2}{3.5^2} + \frac{(1.5)^2(5.4)^2}{(3.5)^3} + \cdots = \frac{5^2}{2} + \frac{(1.5)(0.2)}{2^2}\left[10 + \frac{3(0.2)}{2} + 0.2\right].$$

The following theorem gives the formula to find the sum of arithmetic-geometric progression of power 3.

Theorem 5.7. *Let $t \in [0,\infty)$ and $h \in (0,\infty)$. Then,*

$$\sum_{r=0}^{\infty}\frac{\beta^r}{\alpha^{r+1}}(t+rh)^3 = \frac{t^3}{(\alpha-\beta)} - \frac{3\beta h}{(\alpha-\beta)}\Big(\frac{t^2}{\beta-\alpha} - \frac{2t\beta h}{(\beta-\alpha)^2}$$

$$+\frac{(2\beta^2h^2)}{(\beta-\alpha)^3} - \frac{\beta h^2}{(\beta-\alpha)^2}\Big) - \frac{3\beta h^2}{(\alpha-\beta)}\Big(\frac{t}{\beta-\alpha}\Big) + \frac{\beta h^3}{(\alpha-\beta)^2}. \tag{42}$$

Proof. Since $\Delta_{(\alpha,\beta)(h)}$ is linear, from Definition 3.1,

$$\frac{t^3}{\beta-\alpha} = (h\Delta_{(\alpha,\beta)(h)})^{-1}t^3 + \frac{3h}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}t^2 +$$

$$\frac{3h^2}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}t + \frac{h^3}{\beta-\alpha}(h\Delta_{(\alpha,\beta)(h)})^{-1}(1). \quad (43)$$

Substituting (40), (37) and (35) in (43), we find

$$(h\Delta_{(\alpha,\beta)(h)})^{-1}t^3 = \frac{t_h^{(3)}}{(\beta-\alpha)} - \frac{3(\beta h)t_h^{(2)}}{(\beta-\alpha)^2} + \frac{3^{(2)}(\beta h)^2 t_h^{(1)}}{(\beta-\alpha)^3} - \frac{3^{(3)}(\beta h)^3 t_h^{(0)}}{(\beta-\alpha)^4}. \quad (44)$$

Now the proof follows from (44) and Lemma 5.1.

In general, one can find similar formulas using $\Delta_{(\alpha,\beta)(h)}$ and its inverse. □

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