$\gamma$–OPEN FUNCTION AND $\gamma$–CLOSED FUNCTIONS

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Abstract: In this paper we define two types of functions of topological spaces: $\gamma$–open functions and $\gamma$–closed functions. In addition, we examine the relation of these functions among themselves and their relation with $\gamma$–continuous functions. In the following, we study some properties of $\gamma$–open and $\gamma$–closed functions.

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1. Introduction

The notion of $\gamma$–operation in a topological space and the notion of the $\gamma$–open set were introduced by the Japanese mathematician H. Ogata in 1991. Further, through the notion of the $\gamma$–open set, Ogata defined the $\gamma$–$T_i$ $(i = 0, \frac{1}{2}, 1, 2)$ spaces.

In 1992 F.U. Rehman and B. Ahmad defined the $\gamma$–interior, $\gamma$–exterior, $\gamma$–closure and $\gamma$–boundary of a subset of a topological space (see [9]).

In 2003 B. Ahmad and S. Hussain studied many properties of a $\gamma$–operation in a topological space, they defined the meaning of $\gamma$–neighborhood as well as $\gamma$–neighborhood of the base (see [1]).

In 2009 C.K. Basu, B.M. Uzzal Afsan and M.K. Chash defined $\gamma$–continuity of function (see [3]).

In our previous work [4]: “$\gamma^*$–operation in the product of topological...
spaces”, we have defined the concept of $\gamma$-operation in the topological product of topological spaces $(X_i, T_i), i \in I$, through $\gamma_i$-operations of spaces $(X_i, T_i), i \in I$.

As noted before, in this paper we define two types of functions of topological spaces: $\gamma$-open functions and $\gamma$-closed functions and also we study some properties of these functions.

2. Preliminaries

Definition 1. ([11]) Let $(X, T)$ be the topological space. The function $\gamma : T \to P(X), (P(X), \gamma(G),$ is the partition of $X)$, such that for each $G \in T, G \subseteq \gamma(G)$, is called $\gamma$-operation in the topological space $(X, T)$.

Examples of $\gamma$-operation in a topological space $(X, T)$, are functions: $\gamma : T \to P(X)$, given by: $\gamma(G) = G, \gamma(G) = clG, \gamma(G) = int(clG)$, etc.

Definition 2. ([11]) Let $(X, T)$ be the topological space and $\gamma : T \to P(X)$ a $\gamma$-operation in this space. The set $G \subseteq X$ is called $\gamma$-open, if for every $x \in G$, there exists the set $U \in T$, such that $x \in U \subseteq \gamma(U) \subseteq G$.

The family of all $\gamma$-open sets of topological space $(X, T)$, is denoted by: $T_\gamma$. This means:

$$T_\gamma = \{G \subseteq X : \forall x \in G, \exists U \in T : x \in U \subseteq \gamma(U) \subseteq G\}. \quad (1)$$

Definition 3. ([2]) The set $F$, is called $\gamma$-closed, if its complement $F^C = X \setminus F$ is the $\gamma$-open set in $X$.

By Definition 3, it turns out that if the set $F$ is $\gamma$-closed, it is closed set. True, if $F$ is $\gamma$-closed, then $F^C = X \setminus F$ is a $\gamma$-open set and as such is open set. It means $F$ is closed set. By the definition of $\gamma$-operation, $\gamma$-open set and $\gamma$-closed set some simple statements are derived, which are given by the following theorem.

Theorem 4. ([2]) Let $(X, T)$ be the topological space and $\gamma : T \to P(X)$ a $\gamma$-operation in this space, also let be $T_\gamma$ as given in equation (1), then:

1. $T_\gamma \subseteq T$
2. $\emptyset, X \in T_\gamma$ ($\emptyset$ and $X$ are $\gamma$–closed sets),

3. If $\gamma(G) = G, \forall G \in T$, then $T_\gamma = T$,

4. If $T = \{\emptyset, X\}$ (in discrete topology), then for each $\gamma$–operation, the only $\gamma$–open and $\gamma$–closed sets are $\emptyset$ and $X$.

**Proof.**

1. From $G \in T_\gamma$ it follows that for every $x \in G$, exists $U_x \in T$ such that $x \in U_x \subseteq \gamma(U_x) \subseteq G$, so $G = \cup \{U_x : x \in G\} \in T$ (as a union of sets $U_x \in T$). This means $T_\gamma \subseteq T$.

2. Since $\emptyset$ has no element, we can consider that each “element” satisfies the condition of Definition 1. It means $\emptyset \in T_\gamma$. On the other hand, $\forall x \in X$, there exists an open set $U$ (at least set $X$), so $x \in U$ and $\gamma(U) \subseteq X$. It means $X \in T_\gamma$.

3. Let $\gamma(G) = G, \forall G \in T$. From $G \in T$ and $x \in G$ we have: $x \in G$ and $\gamma(G) = G \subseteq G$, it means $T \subseteq T_\gamma$. From statement 1. it follows that $T_\gamma = T$.

4. It follows directly from the statements 1. and 2.

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**Theorem 5.** ([1]) Let $(X, T)$ be the topological space and $\gamma : T \to \mathcal{P}(X)$ a $\gamma$–operation in this space, then:

1. $T_\gamma \subseteq T$,

2. $\emptyset, X \in T_\gamma$ ($\emptyset$ and $X$ are $\gamma$–closed sets).

**Proof.**

1. Let $x \in \cup_{i \in I} G_i (G_i \in T_\gamma)$. Then, there exists $i_0 \in I$, such that $x \in G_{i_0}$. Since $G_{i_0} \in T_\gamma$, there exists $G \in T$ such as $x \in G$ and $\gamma(G) \subseteq G_{i_0} \subseteq \cup_{i \in I} G_i$. It means $\cup_{i \in I} G_i \in T_\gamma$.

2. That the intersection of the two $\gamma$–open sets is not always a $\gamma$–open set, as shown by the following example.

**Example 1.** Let $X = \{a, b, c\}$ with the topology

$$T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}.$$
In $X$, we define $\gamma$–operation in this way:

$$\gamma(G) = \begin{cases} G, & b \in G \\ \text{cl}G, & b \notin G \end{cases}. \quad (2)$$

We define the $T$ family, by proving which of the sets from the $T$ family, satisfy the terms of Definition 2.

We consider the set $\{a\}$. Sets $\{a\}, \{a, b\}$ and $\{a, c\}$, are open sets that contain the single element $a$, of the set $\{a\}$. On the other hand, we have:

$$\gamma(\{a\}) = \text{cl}\{a\} = \{a, c\} \not\subseteq \{a\}, \quad \gamma(\{a, b\}) = \{a, b\} \not\subseteq \{a\}, \quad \gamma(\{a, c\}) = \{a, c\} \not\subseteq \{a\}.$$  

It means $\{a\} \not\in T$.  

Similarly, proving to all open sets of given space, we find that:

$$T = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}. \quad (3)$$

We notice that $\{a, b\} \in T$ and $\{a, c\} \in T$, but $\{a, b\} \cap \{a, c\} = \{a\} \not\in T$.

**Remark 1.** The statement 2. of Theorem 4 shows that $T$ is not always the topology in $X$.

Based on Theorem 5 and De-Morgan laws, it follows:

**Theorem 6.** Let $(X, T)$ be the topological space and $\gamma : T \to P(X)$ a $\gamma$–operation in this space, then:

1. If for any $i \in I, F_i$ is the $\gamma$–closed set in $X$, then $\cap_{i \in I} F_i$ is the $\gamma$–closed set in $X$,

2. If $F_1, F_2$ are $\gamma$–closed sets in $X$, then $F_1 \cup F_2$, is not always $\gamma$–closed set.

The statement 2. of Theorem 6 is proved by Example 1, according to which $\{c\}^c = \{a, b\} \in T$ and $\{b\}^c = \{a, c\} \in T$. So, sets $\{c\}$ and $\{b\}$, are $\gamma$–closed, but their union $\{b, c\}$, is not $\gamma$–closed because $\{b, c\}^c = \{a\} \not\in T$.

**Definition 7.** ([11]) Let $(X, T)$ be the topological space and $A \subseteq X$. The point $x \in A$ is called the point of $\gamma$–interior of set $A$, if there exists an open neighborhood $U$ of point $x$, so $\gamma(U) \subseteq A$. The set of all $\gamma$–interiors points of set $A$ is called $\gamma$–interior and is symbolically denoted by $\text{int}_\gamma(A)$. This means:

$$\text{int}_\gamma(A) = \{x \in A : x \in U \in T, \gamma(U) \subseteq A\} \subseteq A.$$  

By Definition 7, it turns out that $\text{int}_\gamma(A) \subseteq \text{int}A$. Also from Definitions 2 and 7, it turns out that $A$ is a $\gamma$–open, then and only when $\text{int}_\gamma(A) = \text{int}A$. 

Definition 8. ([11]) Let \((X, \mathcal{T})\) be the topological space and \(A \subseteq X\). The point \(x \in A\) is called the point of \(\gamma\)-closure of set \(A\), if \(\gamma(U) \cap A \neq \emptyset\) for each open neighborhood \(U\) of point \(x\). The set of all \(\gamma\)-closures points of set \(A\) is called \(\gamma\)-closure and is symbolically marked by \(\text{cl}_\gamma(A)\). It means:

\[
\text{cl}_\gamma(A) = \{ x \in A : \gamma(U) \cap A \neq \emptyset, \text{ for each open neighborhood } U \text{ of point } x \}.
\]

By Definition 8, it turns out that \(clA \subseteq cl_\gamma(A)\) and \(A \subseteq cl_\gamma(A)\). Also from Definitions 3 and 8, it turns out that \(A\) is a \(\gamma\)-closed, then and only when \(A = cl_\gamma(A)\).

Definition 9. ([1]) The set \(U \subseteq X\) is called the \(\gamma\)-neighborhood of point \(x \in X\), if there exists the \(\gamma\)-open set \(V\), such that \(x \in V \subseteq U\).

Definition 10. ([3]) Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}')\) be two topological space and \(\gamma : \mathcal{T} \rightarrow \mathcal{P}(X)\) a \(\gamma\)-operation in \((X, \mathcal{T})\). The function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')\) is called the \(\gamma\)-continuous if the inverse image of any \(V\)-open set in \(Y\) is the \(\gamma\)-open set in \(X\).

Definition 11. ([5]) Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}')\) be two topological space. The function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')\) is called open if the image \(f(G)\) of each open set \(G\) in \(X\) is open set in \(Y\).

Definition 12. ([5]) Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}')\) be two topological space. The function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')\) is called closed if the image \(f(F)\) of each open set \(F\) in \(X\) is open set in \(Y\).

3. Main results

3.1. \(\gamma\)-open functions and \(\gamma\)-closed functions

Definition 13. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}')\) be two topological space and \(\gamma\) a \(\gamma\)-operation in space \(Y\). The function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')\) is called the \(\gamma\)-open function if the image \(f(G)\) of any \(G\) open set in \(X\) is the \(\gamma\)-open set in \(Y\).
Definition 14. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}^{'})\) be two topological space and \(\gamma\) a \(\gamma\)–operation in space \(Y\). The function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})\) is called the \(\gamma\)–closed function if the image \(f(F)\) of any \(F\) closed set in \(X\) is the \(\gamma\)–closed set in \(Y\).

By Definitions 13 and 14, the following simple statement follows:

Statement 1. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}^{'})\) be two topological space and \(\gamma\) a \(\gamma\)–operation in space \(Y\).

1. If the function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})\) is \(\gamma\)–open, then it is also open;

2. If the function \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})\) is \(\gamma\)–closed, then it is also closed.

Proof. 1. Let \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})\) be \(\gamma\)–open function and \(G\) open set in \(X\). Then from Definition 13, \(f(G)\) is \(\gamma\)–open set in \(Y\) and as such is open set in \(Y\). It means \(f\) is an open function.

2. Let \(f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})\) be \(\gamma\)–open function and \(F\) closed set in \(X\). Then from Definition 14, \(f(F)\) is \(\gamma\)–closed set in \(Y\) and as such is open set in \(Y\). It means \(f\) is a closed function.

Next, by means of different examples we will see what is the relation among the \(\gamma\)–open, \(\gamma\)–closed and \(\gamma\)–continuous functions.

Example 2. Let be \(X = \{a, b, c\}\) with the topology \(\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}\) and \(\gamma\)–operation in \(X\), given with: \(\gamma(G) = clG, \forall G \in \mathcal{T}\). The closed sets in \(X\) are: \(\emptyset, \{b, c\}, \{a, b\}, \{c\}, \{b\}\) and \(X\). By definition of \(\gamma\)–operation we have:

\[
\begin{align*}
\gamma(\emptyset) &= cl\emptyset = \emptyset, \\
\gamma(\{a\}) &= cl\{a\} = \{a\}, \\
\gamma(\{c\}) &= cl\{c\} = \{c\}, \\
\gamma(\{a, b\}) &= cl\{a, b\} = \{a, b\}, \\
\gamma(\{a, c\}) &= cl\{a, c\} = X, \\
\gamma(X) &= X.
\end{align*}
\]

We will consider which of the family members \(\mathcal{T}\) meet the terms of Definition 1. For example the set \(\{a\}\) is not \(\gamma\)–open because open sets that contain its only element a are: \(\{a\}, \{a, b\}, \{a, c\}\), and \(X\), but none of the sets \(\gamma(\{a\}), \gamma(\{a, b\}), \gamma(\{a, c\})\) and \(\gamma(X)\) do not contained in \(\{a\}\). By proving equally to all open sets in \(X\), we conclude that \(\mathcal{T}_\gamma = \{\emptyset, \{c\}, \{a, b\}, X\}\).

Let be \(Y = \{1, 2\}\) with the topology \(\mathcal{T}^{'\prime} = \{\emptyset, \{1\}, Y\}\). In space \(Y\) let \(\gamma^{'\prime}\)–operation be given in this way: \(\gamma^{'\prime}(G) = G, \forall G \in \mathcal{T}^{'\prime}\). Since \(\gamma^{'\prime}(G) = G, \forall G \in \mathcal{T}^{'\prime}\)...
it comes out that $T'_\gamma = T' = \emptyset, \{ 1 \}, Y$. We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x \in \{ a, c \} \\ 2, x \in \{ b \} \end{cases}.$$ 

From the definition of function $f$ we have:

$$f(\emptyset) = \emptyset, f(\{ a \}) = \{ 1 \}, f(\{ c \}) = \{ 1 \}, f(\{ a, b \}) = \{ 1, 2 \} = Y,$$

$$f(\{ a, c \}) = \{ 1 \}, f(X) = Y.$$

It means that the image of any open set in $X$, is the $\gamma'$-open set in $Y$, so $f$ is the $\gamma'$-open function.

Note that $\{ c \}$ is a closed set in $X$, because $\{ c \}^c = \{ a, b \} \in T$, but $f(\{ c \}) = \{ 1 \}$ is not a $\gamma'$-closed set in $Y$, so $f$ is not $\gamma'$-closed function.

We also note that $\{ 1 \}$ is $\gamma'$-open in $Y$, but $f^{-1}(\{ 1 \}) = \{ a, c \} \notin T_\gamma$. It means that $f$ is not a $\gamma$-continuous function.

**Example 3.** Let be $X = \{ a, b, c \}$ with the topology $T = \emptyset, \{ a \}, \{ c \}, \{ a, b \}, \{ a, c \}, X$ and let be given $\gamma$-operation in $X$, with: $\gamma(G) = cl_G, \forall G \in T$ (see Example 2). We have $T_\gamma = \emptyset, \{ c \}, \{ a, b \}, X$.

Let be $Y = \{ 1, 2, 3 \}$ with the indiscrete topology $T' = \emptyset, Y$. Since the only open sets in $\gamma'$ are $\emptyset$ and $Y$, for whatever $\gamma'$-operation in $Y$ we have $T'_{\gamma'} = \emptyset, Y$.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x = a \\ 2, x = b \\ 3, x = c \end{cases}.$$ 

It is clear that $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. It means that the inverse image of any open set in $Y$, is $\gamma$-open set in $X$. It means that $f$ is the $\gamma$-continuous function.

On the other hand for example $\{ a \}$ is a closed set in $X$, but $f(\{ a \}) = \{ 1 \} \notin T_\gamma'$. It means that $f$ is not $\gamma'$-open function.

As well $\{ b, c \}$ is closed set in $X$, but $f(\{ b, c \}) = \{ 1, 2 \}$ and $\{ 1, 2 \}$ is not $\gamma'$-closed in $Y$. It means that $f$ is not a $\gamma'$-closed function.
**Example 4.** Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}.$$ 

Closed sets in $X$ are: $\emptyset$, $\{b, c\}$, $\{a, c\}$, $\{c\}$ and $X$.

Let be $Y = \{1, 2\}$ with the topology $\mathcal{T}' = \{\emptyset, \{1\}, Y\}$ and let be given $\gamma'$-operation in $Y$ with $\gamma'(G) = G, \forall G \in \mathcal{T}'$. We have $\mathcal{T}_\gamma = \{\emptyset, \{1\}, Y\}$. $\gamma'$-closed sets in $Y$ are: $\emptyset$, $\{2\}$ and $Y$.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, & x \in \{a\} \\ 2, & x \in \{b, c\} \end{cases}.$$ 

We have $f(\emptyset) = \emptyset, f(\{b, c\}) = \{2\}, f(\{a\}) = Y, f(\{c\}) = \{2\}, f(X) = Y$. It means that the image of each closed set in $X$, is $\gamma'$-closed set in $Y$. So $f$ is $\gamma'$-closed function.

On the other hand $\{b\}$ is open set in $X$, but $f(\{b\}) = \{2\}$ is not $\gamma'$-open set in $Y$, consequently $f$ is not $\gamma'$-open function.

**Example 5.** Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, X\}$$

and let be $\gamma(G) = G, \forall G \in \mathcal{T}$. It means $\mathcal{T}_\gamma = \{\emptyset, \{a\}, X\}$. Closed sets in $X$ are: $\emptyset$, $\{b, c\}$ and $X$.

Let be now $Y = \{1, 2\}$ with the topology $\mathcal{T}' = \{\emptyset, \{1\}, Y\}$ and $\gamma'(G) = G, \forall G \in \mathcal{T}'$. It means $\mathcal{T}_\gamma' = \{\emptyset, \{1\}, Y\}$. $\gamma'$-closed sets in $Y$ are: $\emptyset$, $\{2\}, Y$.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, & x \in \{c\} \\ 2, & x \in \{a, b\} \end{cases}.$$ 

We have $f(\emptyset) = \emptyset, f(\{b, c\}) = Y, f(X) = Y$. It means that the image of each closed set in $X$, is the $\gamma'$-closed set in $Y$. So $f$ is $\gamma'$-closed function. On the other hand $\{1\}$ is open set in $Y$, but $f^{-1}(\{1\}) = \{c\}$ which is not a $\gamma$-open set in $X$. It means that $f$ is not a $\gamma$-continuous function.

The last four examples show that the meanings: $\gamma$-open function, $\gamma$-closed function, $\gamma$-continuous function are independent of each other.
3.2. Some properties of $\gamma$-open and $\gamma$-closed functions

The following theorems provide some statements equivalent to Definition 13, respectively Definition 14.

**Theorem 15.** Let $(X,T)$ and $(Y,T')$ be two topological spaces, $\gamma$ a $\gamma$-operation in space $Y$ and $f : X \to Y$. The following statements are equivalent:

1. $f$ is the $\gamma$-open.

2. $\emptyset, X \in T_\gamma$ ($\emptyset$ and $X$ are $\gamma$-closed sets)

3. For each open set $B$ of base in $X$, $f(B)$ is $\gamma$-open set in $Y$.

4. For each $x \in X$ and each neighborhood $U$ of the point $x$ in $X$ there exists $\gamma$-neighborhood $V$ of the point $f(x)$ in $Y$ such that $V \subseteq f(U)$.

**Proof.** (1) $\Rightarrow$ (2) Let $f$ be the $\gamma$-open function. It means that for each open set $G$ in $X$, $f(G)$ is $\gamma$-open in $Y$. From $\text{int}A \subseteq A \Rightarrow f(\text{int}A) \subseteq f(A) \Rightarrow \text{int}_\gamma f(\text{int}A) \subseteq \text{int}_\gamma f(A)$. Since $\text{int}A$ is the open set in $X$ and $f$ the $\gamma$-open function, it turns out that $f(\text{int}A)$ is $\gamma$-open set in $Y$, and hence $\text{int}_\gamma f(\text{int}A) = f(\text{int}A) \Rightarrow f(\text{int}A) \subseteq \text{int}_\gamma f(A)$.

(2) $\Rightarrow$ (3) Let $B$ be the open set of base in $X$. Then $\text{int}B = B \Rightarrow f(B) = f(\text{int}B) \subseteq \text{int}_\gamma f(B) \subseteq f(B)$. It means that $\text{int}_\gamma f(B) = f(B) \Rightarrow f(B)$ is $\gamma$-open in $Y$.

(3) $\Rightarrow$ (4) Let be $x \in X$ and $U$ neighborhood of point $x$ in $X$. Then there exists the open set $G$ in $X$ such that $x \in G \subseteq U$ and there exists an open set of base $B$ such that $x \in B \subseteq G \subseteq U$. Note that $V = f(B)$, according to (3), $V$ is $\gamma$-open in $Y$ and as such is the $\gamma$-neighborhood of its point $f(x) \in V \subseteq f(U)$. ($B \subseteq U \Rightarrow f(B) = V \subseteq f(U)$).

(4) $\Rightarrow$ (1) Let $G$ be open set in $X$. Then $G$ is the neighborhood of each point $x \in G$. According to (4), for every $x \in G$, there exists the $\gamma$-neighborhood $V_x$ of the point $f(x)$ in $Y$, such that $V_x \subseteq f(G)$. From the definition of $\gamma$-neighborhood for every $x \in G$ there exists the $\gamma$-open set $H_x$ in $Y$, such that $f(x) \in H_x \subseteq V_x \subseteq f(G)$. This means $f(G) = f(\bigcup_{x \in G}\{x\}) = \bigcup_{x \in G}\{f(x)\} \subseteq \bigcup_{x \in G}H_x \subseteq f(G)$. It means $f(G) = \bigcup_{x \in G}H_x$. So $f(G)$ is the $\gamma$-open set as the union of the $\gamma$-open sets. Finally $f$ is the $\gamma$-open function.
Theorem 16. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}^{'})\) be two topological spaces and \(\gamma\) a \(\gamma\)-operation in space \(Y\) and \(f : X \rightarrow Y\). The function \(f : X \rightarrow Y\) is \(\gamma\)-closed if and only if for each \(A \subseteq X\) is valid: \(cl_Y f(A) \subseteq f(cl A)\).

Proof. Let \(f : X \rightarrow Y\) be the \(\gamma\)-closed function. Then \(f(cl A)\) is \(\gamma\)-closed set in \(Y\), because \(cl A\) is closed set in \(X\). From \(A \subseteq cl A \Rightarrow f(A) \subseteq f(cl A) \Rightarrow cl_{\gamma} f(A) \subseteq cl_{\gamma} f(cl A) = f(cl A)\), because \(f(cl A)\) is \(\gamma\)-closed in \(X\). It means \(cl_{\gamma} f(A) \subseteq f(cl A)\).

Conversely: For each \(A \subseteq X\), let \(cl_{\gamma} f(A) \subseteq f(cl A)\) and let be \(F \subseteq X\) the closed set in \(X\). From the assumption \(f(F) \subseteq cl_{\gamma} f(F) \subseteq f(cl F) = f(F)\), that because \(f\) is closed at \(X\). It means \(f(F) = cl_{\gamma} f(F)\). So \(f(F)\) is the \(\gamma\)-closed set in \(Y\) and consequently \(f\) is the \(\gamma\)-closed function. \(\Box\)

Theorem 17. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}^{'})\) be two topological spaces and \(\gamma\) a \(\gamma\)-operation in space \(Y\). The function \(f : X \rightarrow Y\) is \(\gamma\)-closed if and only if for each \(B \subseteq Y\) and any open set \(G \subseteq X\) such that \(f^{-1}(B) \subseteq G\), there exists the \(\gamma\)-open set \(U\) in \(Y\), such that \(B \subseteq U\) and \(f^{-1}(U) \subseteq G\).

Proof. Let \(f : X \rightarrow Y\) be the \(\gamma\)-closed function. Let be further \(B \subseteq Y\) and \(G \subseteq X\) the open set in \(X\), such that \(f^{-1}(B) \subseteq G\). Then \(f(X \setminus G)\) is \(\gamma\)-closed set in \(Y\), because \(X \setminus G\) is closed in \(X\) and \(f\) is \(\gamma\)-closed. The set \(U = Y \setminus f(X \setminus G)\) is \(\gamma\)-open in \(Y\). Since \(f^{-1}(B) \subseteq G \Rightarrow X \setminus G \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \Rightarrow f(X \setminus G) \subseteq f(f^{-1}(Y \setminus B)) \subseteq Y \setminus B \Rightarrow Y \setminus f(X \setminus G) \supseteq Y \setminus (Y \setminus B) = B\). It means \(B \subseteq U\).

As well \(f^{-1}(U) = f^{-1}(Y \setminus f(X \setminus G)) = X \setminus f^{-1}(f(X \setminus G)) \subseteq X \setminus (X \setminus G) = G\). It means \(f^{-1}(U) \subseteq G\) and \(B \subseteq U\).

Conversely: Let \(F\) be any closed set in \(X\). We have to prove that \(f(F)\) is \(\gamma\)-closed in \(Y\). The set \(B = Y \setminus f(F)\) satisfies the condition \(f^{-1}(B) = f^{-1}(Y \setminus f(F)) = X \setminus f^{-1}(f(F)) \subseteq X \setminus f\). Note \(X \setminus F = G\). \(G\) is open set in \(X\) such that \(f^{-1}(B) \subseteq G\). From the assumption exists the \(\gamma\)-open set \(U\) in \(Y\), such that \(B \subseteq U\) and \(f^{-1}(U) \subseteq G\). It means \(B = Y \setminus f(F) \subseteq U\) and \(f^{-1}(U) \cap (X \setminus G) = \emptyset \Rightarrow f(f^{-1}(U) \cap (X \setminus G)) = f(\emptyset) = \emptyset \Rightarrow U \cap f(X \setminus G) = \emptyset\). That is, \(U \subseteq Y \setminus f(X \setminus G) = Y \setminus f(F)\). So we have \(Y \setminus f(F) \subseteq U \subseteq Y \setminus f(F)\).

That is, \(U = Y \setminus f(F) \Rightarrow f(F) = Y \setminus U\) is \(\gamma\)-closed set in \(Y\), because \(U\) is \(\gamma\)-open in \(Y\). Thus \(f\) is \(\gamma\)-closed. \(\Box\)

Theorem 18. Let \((X, \mathcal{T})\) and \((Y, \mathcal{T}^{'})\) be two topological spaces and \(\gamma\) a \(\gamma\)-operation in space \(Y\). The function \(f : X \rightarrow Y\) is \(\gamma\)-open if and only if for each \(B \subseteq Y\) and any open set \(F \subseteq X\) such that \(f^{-1}(B) \subseteq F\), there exists the \(\gamma\)-closed set \(S\) in \(Y\), such that \(B \subseteq S\) and \(f^{-1}(S) \subseteq F\).
Proof. Let \( f : X \to Y \) be the \( \gamma \)-open function. Let be further \( B \subseteq Y \) and \( F \subseteq X \) the closed set in \( X \), such that \( f^{-1}(B) \subseteq F \). Then \( f(X \setminus F) \) is \( \gamma \)-open set in \( Y \), because \( X \setminus F \) is open in \( X \) and \( f \) is \( \gamma \)-open. The set \( S = Y \setminus f(X \setminus F) \) is \( \gamma \)-closed in \( Y \). Since \( f^{-1}(B) \subseteq F \Rightarrow X \setminus F \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \Rightarrow f(X \setminus F) \subseteq f(f^{-1}(Y \setminus B)) \subseteq Y \setminus f(X \setminus F) \supseteq Y \setminus f(X \setminus F) = B \). It means \( B \subseteq S \). As well \( f^{-1}(S) = f^{-1}(Y \setminus f(X \setminus F)) = X \setminus f^{-1}(f(X \setminus F)) \subseteq X \setminus (X \setminus F) = F \). It means \( f^{-1}(S) \subseteq F \) and \( B \subseteq S \).

Conversely: Let \( G \) be any open set in \( X \). We have to prove that \( f(G) \) is \( \gamma \)-open in \( Y \). The set \( B = Y \setminus f(G) \) satisfies the condition \( f^{-1}(B) = f^{-1}(Y \setminus f(G)) = X \setminus f^{-1}(f(G)) \subseteq X \setminus G \). Note \( X \setminus G = F \). \( F \) is closed set in \( X \) such that \( f^{-1}(B) \subseteq F \). From the assumption exists the \( \gamma \)-closed set \( S \) in \( Y \), such that \( B \subseteq S \) and \( f^{-1}(S) \subseteq F \). It means \( B = Y \setminus f(G) \subseteq S \) and \( f^{-1}(S) \cap (X \setminus F) = \emptyset \Rightarrow f^{-1}(S) \cap (X \setminus F) = f(\emptyset) = \emptyset \Rightarrow S \cap f(X \setminus F) = \emptyset \).

It means \( S \subseteq Y \setminus f(X \setminus F) = Y \setminus f(G) \). So we have \( Y \setminus f(G) \subseteq S \subseteq Y \setminus f(G) \). It means \( S = Y \setminus f(G) \Rightarrow f(F) = Y \setminus S \) is \( \gamma \)-open set in \( Y \), because \( S \) is \( \gamma \)-closed in \( Y \). Thus \( f \) is \( \gamma \)-open.

The following theorems are related to the composition of \( \gamma \)-open (\( \gamma \)-closed) functions.

**Theorem 19.** Let \((X, \mathcal{T}_1), (Y, \mathcal{T}_2)\) and \((Z, \mathcal{T}_3)\) be topological spaces and \( f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)\) the functions of topological spaces. Let be further \( \gamma_2 \) and \( \gamma_3 \), the \( \gamma \)-operations in \( Y \) and \( Z \) spaces, respectively. Then

1. If \( f \) and \( g \) are \( \gamma_2 \), respectively \( \gamma_3 \)-open functions, then their composition \( g \circ f : X \to Z \) is \( \gamma_3 \)-open function.
2. If \( f \) and \( g \) are \( \gamma_2 \), respectively \( \gamma_3 \)-closed functions, then their composition \( g \circ f : X \to Z \) is \( \gamma_3 \)-closed function.
3. If \( f \) is open function and \( g \) is \( \gamma_3 \)-open functions, then their composition \( g \circ f : X \to Z \) is \( \gamma_3 \)-open function.
4. If \( f \) is closed function and \( g \) is \( \gamma_3 \)-closed functions, then their composition \( g \circ f : X \to Z \) is \( \gamma_3 \)-closed function.

Proof. 1. Let \( f \) and \( g \) be \( \gamma_2 \), respectively \( \gamma_3 \)-open functions and let \( G \) be any open set in \( X \). Then from \((g \circ f)(G) = g(f(G))\) and since \( f \) is \( \gamma_2 \)-open, it follows that \( f(G) \) is \( \gamma_2 \)-open set in \( Y \) and as such it is open.
set in $Y$. As $g$ is $\gamma_3$–open function it follows that $(g \circ f)(G) = g(f(G))$ is $\gamma_3$–open set in $Z$. So the composition $g \circ f : X \to Z$ is $\gamma_3$–open function.

2. Let $f$ and $g$ be $\gamma_2$, respectively $\gamma_3$–closed functions and let $F$ be any closed set in $X$. Then from $(g \circ f)(F) = g(f(F))$ and since $f$ is $\gamma_2$–closed, it follows that $f(F)$ is $\gamma_2$–closed set in $Y$ and as such it is closed set in $Y$. As $g$ is $\gamma_3$–closed set in $Y$. As $g$ is $\gamma_3$–closed function it follows that $(g \circ f)(G) = g(f(G))$ is $\gamma_3$–closed set in $Z$. So the composition $g \circ f : X \to Z$ is $\gamma_3$–closed function.

3. Let $f$ be any open function and $g$ be $\gamma_3$–open functions and $G$ let be any open set in $X$. Then from $(g \circ f)(G) = g(f(G))$ and since $G$ is open set in $X$ and $f$ open function, it follows that $f(G)$ is open set in $Y$. As $g$ is $\gamma_3$–open function it follows that $(g \circ f)(G) = g(f(G))$ is $\gamma_3$–open set in $Z$. It means $g \circ f : X \to Z$ is $\gamma_3$–open function.

4. Let $f$ be any closed function and $g$ be $\gamma_3$–closed functions and $F$ let be any closed set in $X$. Then from $(g \circ f)(G) = g(f(G))$ and since $F$ is closed set in $X$ and $f$ closed function, it follows that $f(F)$ is closed set in $Y$. As $g$ is $\gamma_3$–closed function it follows that $(g \circ f)(G) = g(f(G))$ is $\gamma_3$–closed set in $Z$. It means $g \circ f : X \to Z$ is $\gamma_3$–closed function.

\[ \Box \]

**Theorem 20.** Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ the functions of topological spaces. Let be further $\gamma_3$ a $\gamma$–operations in $Z$ spaces. Then

1. If the composition $g \circ f : X \to Z$ is $\gamma_3$–open function and $f$ is the surjective and continuous function, then the function $g$ is $\gamma_3$–open.

2. If the composition $g \circ f : X \to Z$ is $\gamma_3$–closed function and $f$ is the surjective and continuous function, then the function $g$ is $\gamma_3$–closed.

**Proof.** 1. Let $G$ be any open set in $Y$ and $f$ the surjective and continuous function. Since $f$ is surjective, there is a set $H \subseteq X$ such that $f^{-1}(G) = H$. From the continuity of the function $f$ it follows that $H$ is the open set in $X$. Since $g \circ f : X \to Z$ is $\gamma_3$–open, then $(g \circ f)(H)$ is $\gamma_3$–open in $Z$. Further $(g \circ f)(H) = g(f(H)) = g(f(f^{-1}(G))) = (\text{since } f \text{ is a surjective}) = g(G)$. It means $g(G)$ is $\gamma_3$–open set and consequently $g$ is the $\gamma_3$–open function.
2. Let $F$ be any closed set in $Y$ and $f$ the surjective and continuous function. Since $f$ is surjective, there is a set $S \subseteq X$ such that $f^{-1}(F) = S$. From the continuity of the function $f$ it follows that $S$ is the closed set in $X$. Since $g \circ f : X \rightarrow Z$ is $\gamma_3$-closed, then $(g \circ f)(S)$ is $\gamma_3$-closed in $Z$. Further $(g \circ f)(S) = g(f(S)) = g(f(f^{-1}(F))) =$ (since $f$ is a surjective) $= g(F)$. It means $g(F)$ is $\gamma_3$-closed set and consequently $g$ is the $\gamma_3$-closed function.

\[\square\]

**Theorem 21.** Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces and $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \rightarrow (Z, \mathcal{T}_3)$ the functions of topological spaces. Let be further $\gamma_2$ and $\gamma_3$ the $\gamma$-operations in $Y$ and $Z$ spaces, respectively. Then

1. If the composition $g \circ f : X \rightarrow Z$ is $\gamma_3$-open function and $g$ is injective and $\gamma_2$-continuous function, then the function $f$ is $\gamma_2$-open.

2. If the composition $g \circ f : X \rightarrow Z$ is $\gamma_3$-closed function and $g$ is injective and $\gamma_2$-continuous function, then the function $f$ is $\gamma_2$-closed.

**Proof.** 1. Let $G \subseteq X$ be any open set in $X$. As $g \circ f : X \rightarrow Z$ is $\gamma_3$-open function, then $(g \circ f)(G) = g(f(G))$ is $\gamma_3$-open set in $Z$ and as such it is open set in $Z$. From the $\gamma_2$-continuity of the function $g$, it follows that $g^{-1}(g(f(G)))$ is $\gamma_2$-open set in $Y$. As $g$ is injective, it follows that $g^{-1}(g(f(G))) = f(G)$. It means $f(G)$ is $\gamma_2$-open in $Y$. Consequently $f$ is $\gamma_2$-open function.

2. Let $F \subseteq X$ be any closed set in $X$. As $g \circ f : X \rightarrow Z$ is $\gamma_3$-closed function, then $(g \circ f)(G) = g(f(G))$ is $\gamma_3$-closed set in $Z$ and as such it is closed set in $Z$. From the $\gamma_2$-continuity of the function $g$, it follows that $g^{-1}(g(f(F)))$ is $\gamma_2$-closed set in $Y$. As $g$ is injective, it follows that $g^{-1}(g(f(F))) = f(F)$. It means $f(F)$ is $\gamma_2$-closed in $Y$. Consequently, $f$ is $\gamma_2$-closed function.

\[\square\]

**Theorem 22.** 1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $\gamma$ a $\gamma$-operation in space $Y$. Further let $A$ be open subspace in $X$. If the function $f : X \rightarrow Y$ is the $\gamma$-open function then also the retract $f|_A : A \rightarrow Y$ is $\gamma$-open function.

2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $\gamma$ a $\gamma$-operation in space
Y. Further let $A$ be closed subspace in $X$. If the function $f : X \to Y$ is the $\gamma$–closed function then also the retract $f_{|A} : A \to Y$ is $\gamma$–closed function.

Proof. 1. Let $G \subseteq A$ be any open set in $A$. As $A$ is open in $X$ it follows that $G$ is open in $X$. As $f$ is $\gamma$–open function, then $f(G)$ is $\gamma$–open set in $Y$ and $f_{|A}(G) = f(G)$. It means $f_{|A}(G)$ is $\gamma$–open set in $Y$. Consequently $f_{|A} : A \to Y$ is $\gamma$–open function.

2. Let $F \subseteq A$ be any open set in $A$. As $A$ is closed in $X$ it follows that $F$ is closed in $X$. As $f$ is $\gamma$–closed function, then $f(F)$ is $\gamma$–closed set in $Y$ and $f_{|A}(F) = f(F)$. It means $f_{|A}(F)$ is $\gamma$–closed set in $Y$. Consequently $f_{|A} : A \to Y$ is $\gamma$–closed function.

The following theorem shows in which case: $\gamma$–open function and $\gamma$–closed function are equivalent.

**Theorem 23.** Let $(X, T)$ and $(Y, T')$ be topological spaces and $\gamma$ a $\gamma$–operation in space $Y$. Further let $f : X \to Y$ be the bijective function of topological spaces. Then the following statements are equivalent:

1. $f$ is $\gamma$–open function.
2. $f$ is $\gamma$–closed function.

Proof. $(1) \Rightarrow (2)$ Let $f$ be the $\gamma$–open function and $F$ any closed set in $X$. Then $X \setminus F$ is open set in $X$, from where $f(X \setminus F)$ is $\gamma$–open set in $Y$. As $f$ is bijective, we have: $f(X \setminus F) = Y \setminus f(F)$. That is, $Y \setminus f(F)$ is $\gamma$–open in $Y$, and then $f(F)$ is $\gamma$–closed in $Y$. Consequently $f$ is $\gamma$–closed function. $(2) \Rightarrow (1)$ Let $f$ be the $\gamma$–closed function and $G$ any open set in $X$. Then $X \setminus G$ is closed set in $X$, from where $f(X \setminus G)$ is $\gamma$–closed set in $Y$. As $f$ is bijective, we have: $f(X \setminus G) = Y \setminus f(G)$. That is, $Y \setminus f(G)$ is $\gamma$–closed in $Y$, and then $f(G)$ is $\gamma$–open in $Y$. Consequently $f$ is $\gamma$–open function.

References


