HIGHER ORDER ESTIMATION OF THE SOLUTION OF QUASI-INVERSE PROBLEM

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Abstract: In [1, 2] the authors established a priori estimate for the solution of quasi-inverse problem of the same order but for different weight functions. In this paper we establish a priori estimate for a higher order of the same problem, such a problem play an important role in optimal control theory.

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1. Introduction

In [2, 1] the authors considered the following problem

\[
\frac{\partial U_\epsilon}{\partial t} - \frac{\partial^2 U_\epsilon}{\partial x^2} - \epsilon \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} = 0, \quad (x, t) \in Q, \quad t < T,
\]

\[
U_\epsilon(x, T) = X(x), \quad 0 \leq x \leq 1,
\]

\[
U_\epsilon(0, t) = 0, \quad \int_0^1 U_\epsilon(x, t)dx = 0, \quad 0 \leq t \leq T. \quad (1)
\]

A priori estimates for the solution of a quasi-inverse problem with different weight functions were proposed there. Here we establish a priori estimate of
higher order for the above problem (1). We start from the following identity:
\[- \int_{Q_T} ((1 - x)^2 + 2(1 - x)y)(\frac{\partial U_\epsilon}{\partial t} - \frac{\partial^2 U_\epsilon}{\partial x^2} - \epsilon \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t}) (\frac{\partial^3 U_\epsilon}{\partial x^2 \partial t}) dxdt = 0. \quad (2)\]

Integrating by parts, we establish the following equation
\[- \int_0^\tau \int_0^1 ((1 - x)^2 + 2(1 - x)y) \frac{\partial U_\epsilon}{\partial x} \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} dxdt = \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 dxdt + 2 \int_0^\tau \int_0^1 \left( \frac{\partial U_\epsilon}{\partial t} \right)^2 dxdt, \quad (3)\]

\[- \int_0^\tau \int_0^1 ((1 - x)^2 + 2(1 - x)y) \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} dxdt = \int_0^\tau \int_0^1 (1 - x)^2 \frac{\partial^2 U_\epsilon}{\partial x^2} \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} dxdt, \quad (4)\]

\[- \int_0^\tau \int_0^1 ((1 - x)^2 + 2(1 - x)y) \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} dxdt = \epsilon \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial U_\epsilon}{\partial t} \right)^2 dxdt + \epsilon \int_0^\tau \int_0^1 \left( y \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 dxdt. \quad (5)\]

Thus, from (2), we have
\[- \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 dxdt + 2 \int_0^\tau \int_0^1 \left( \frac{\partial U_\epsilon}{\partial t} \right)^2 dxdt + \epsilon \int_0^\tau \int_0^1 \left( y \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 dxdt = \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 dxdt + \epsilon \int_0^\tau \int_0^1 \left( y \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 dxdt. \quad (6)\]

Using theorems about geometric and arithmetic mean inequalities, we estimate the integral on the right hand side through those integrals in the left hand side,
\[- \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 dxdt \leq \frac{\delta_1}{2} \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 dxdt + \frac{1}{2\delta_1} \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial U_\epsilon}{\partial t} \right)^2 dxdt, \]
\[
2 \int_0^\tau \int_0^1 (1 - x) y \frac{\partial^2 U_\epsilon}{\partial x^2} \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \, dx \, dt \\
\leq \delta_2 \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt + \frac{1}{\delta_2} \int_0^\tau \int_0^1 \left( \frac{y \partial^2 U_\epsilon}{\partial x^2} \right)^2 \, dx \, dt.
\]

Choosing \( \delta_1 = \frac{\epsilon}{2} \) and \( \delta_2 = \frac{\epsilon}{4} \) and using formula (10) from [2], we get
\[
\int_0^\tau \int_0^1 \left( \frac{y \partial^2 U_\epsilon}{\partial x^2} \right)^2 \, dx \, dt \leq 4 \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 \, dx \, dt,
\]
and from (6) we have
\[
\int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x} \right)^2 \, dx \, dt + \frac{\epsilon}{2} \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt + \epsilon \int_0^\tau \int_0^1 \left( \frac{y \partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt \\
\leq \frac{17}{\epsilon} \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 \, dx \, dt.
\]

Then we use the identity
\[
\int_0^\tau \left( \frac{\partial U_\epsilon}{\partial x} \right)^2 \, dt = \int_0^\tau dt \int_0^t \frac{\partial}{\partial \eta} \left( \frac{\partial U_\epsilon}{\partial x} \right)^2 \, d\eta + \int_0^\tau \left( \frac{\partial U_\epsilon}{\partial x} \right)^2 \bigg|_{t=0} dt.
\]

We estimate the right hand side of (7) as follows:
\[
\frac{17}{\epsilon} \int_0^\tau \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 \, dx \, dt \\
\leq \frac{68T}{\epsilon} \int_0^\tau dt \int_0^t \int_0^1 (1 - x)^2 \left( \frac{\partial^3 U_\epsilon}{\partial x^2 \partial \eta} \right)^2 \, dx \, d\eta + \frac{34T}{\epsilon} \int_0^1 (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x^2} \right)^2 \bigg|_{t=0} dx.
\]

Then using the Gronwall inequality, we get
\[
\int_Q (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x \partial t} \right)^2 \, dx \, dt + 2 \int_Q \left( \frac{\partial U_\epsilon}{\partial t} \right)^2 \, dx \, dt \\
+ \frac{\epsilon}{2} \int_Q (1 - x)^2 \left( \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt + \epsilon \int_Q \left( \frac{y \partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt \\
\leq \frac{34T}{\epsilon} e^{\frac{136T^2}{\epsilon^2}} \int_0^1 (1 - x)^2 \left( X''(x) \right)^2 \, dx.
\]
Or we can rewrite our result in the following form:

\[
\int_{Q} (1 - x)^2 \left( \frac{\partial^2 U_\epsilon}{\partial x \partial t} \right)^2 \, dx \, dt + \frac{\epsilon}{2} \int_{Q} (1 - x)^2 \left( \frac{\partial^3 U_\epsilon}{\partial x^2 \partial t} \right)^2 \, dx \, dt \\
\leq \frac{34\tau}{\epsilon} e^{\frac{136\epsilon^2}{\tau^2}} \int_{0}^{1} (1 - x)^2 \left( X''(x) \right)^2 \, dx.
\]

References
