IMPROVING RESULTS FOR CUT AND
OPERATOR NORMS ON GRAPHON

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Abstract: Graphon theory has recently begun attracting interdisciplinary research. Although the theory includes many intriguing concepts, one important aspect we often employ in network analysis is the relationship between the cut norm and operator norm of a graphon as an operator on some function spaces. This relationship is well known in the past arguments. However, the authors of the past works restricted the domain of a graphon to $L_\infty(I)$. In this note, we discuss the relationship between the cut norm and operator norm of a graphon in more general situations. We improve the well-known existing inequality and enhance the accuracy of some lemma proofs.

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1. Introduction

Accompanied by the recent development of large-scale graph analysis, graphon and digraphon are attracting the interest of researchers in many fields. Originally, graphon and digraphon were developed as limits of dense-graph sequences under exchangeability [3]. On the other hand, regarding graphons and digraphons as kernels that operate on certain function spaces has extensively
promoted their generality, by applying some insight obtained in functional analysis.

Lovász [9] introduced the so-called cut metric to graphons, which is devised on the basis of the cut metric in graph theory. Considering these ideas, many researchers are now using graphons to obtain graph sequences with desirable properties (see, for instance, [1], [8]).

However, from the perspective of functional analysis, the definition of the kernel operator is often too restrictive. For instance, if we like to consider a function in $L^2$, we should estimate the operator norm of a graphon/digraphon that acts on the $L^2$ space. Moreover, we can obtain inequalities more useful than those introduced in the references cited thus far by adding small assumptions on graphons/digraphons. In this note, we discuss the properties of graphons and digraphons as operators and deduce some improvements on the existing inequalities. Our results in this note are improvements on existing results.

2. On past arguments concerning graphon

Graphon is a tool regarded as the limit of exchangeable dense non-directed graph sequence [9]. Arising from the well-known Aldous–Hoover representation theorem [6], it is often used to discuss the continuous limit of some quantities of a dense, large-scale graph.

Graphon is defined as a symmetric integrable function $W(x, y) : I \times I \rightarrow I$: $W(x, y) = W(y, x)$ for $x, y \in I$. Hereafter, we use the notation $I \equiv [0, 1]$. Roughly speaking, a graphon can be considered a generalization of an adjacency matrix of a weighted non-directed, jointly exchangeable graph. It can also be regarded as the continuum limit of a dense graph sequence.

To formulate the graph limit of digraph sequences, we have the concept of digraphon, which is defined as a five tuple of measurable functions $\overline{W} \equiv \{W_{00}, W_{01}, W_{10}, W_{11}, w\}$, $W_{ij} : I \times I \rightarrow I$ and $w : I \rightarrow I$, that satisfies:

(i) $W_{00}(x, y) = W_{00}(y, x)$,

(ii) $W_{11}(x, y) = W_{11}(y, x)$,

(iii) $W_{01}(x, y) = W_{10}(y, x)$,

(iv) $\sum_{i=0}^{1} W_{0i}(x, y) + W_{11}(x, y) = 1.$
3. Notations

Here, we introduce the notation used in this note.

For a complex number $z \in \mathbb{C}$, $\bar{z}$ represents its complex conjugate. In the following, let $\mathcal{G}$ be an arbitrary open set in $\mathbb{R}$ or $\mathbb{R}^2$. $1_\mathcal{G}$ is the identity function that takes the value of 1 only on a set $\mathcal{G}$ and vanishes elsewhere.

$L_2(\mathcal{G})$ means a set of square-integrable functions defined on $\mathcal{G}$, equipped with the norm

$$\|f\|_{L^2(\mathcal{G})} \equiv \left( \int_{\mathcal{G}} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$ 

The inner product in $L_2(\mathcal{G})$ is defined by

$$(f_1, f_2) \equiv \int_{\mathcal{G}} f_1(x) \bar{f}_2(x) \, dx.$$ 

By $\| \cdot \|_{L^p(\mathcal{G})}$, we denote the usual $L_p$-norm with $1 < p \leq +\infty$ on $\mathcal{G}$:

$$\|f\|_{L^p(\mathcal{G})} \equiv \begin{cases} \left( \int_\mathcal{G} |f(x)|^p \, dx \right)^{\frac{1}{p}} & (p \in [1, +\infty)), \\ \text{ess sup}_{x \in \mathcal{G}} |f(x)| & (p = \infty). \end{cases}$$ 

Let $X$ be a Banach space with norm $\| \cdot \|_X$. For a function $f : I \to X$ in general, we say $f$ is of bounded variation if there is a constant $V > 0$ such that, for arbitrary $N \in \mathbb{N}$, $\sum_{i=1}^{N-1} \|f(t_{i+1}) - f(t_i)\| \leq V$ holds whenever $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1$.

For a Banach space $X$ in general, a set of the semilinear form $f$ is called as the adjoint space of $X$ and is denoted as $X^*$. Here $f$ is called semilinear if it satisfies $f(au + bv) = \bar{a}f(u) + \bar{b}f(v)$ for all $u, v \in X$ and constants $a, b \in \mathbb{C}$.

For a linear operator $T$ on a Banach space $X$, the operator norm of $T$ is defined by

$$\|T\| \equiv \sup_{u \in X} \frac{\|Tu\|_Y}{\|u\|_X} = \sup_{\|u\|_X = 1} \|Tu\|_Y.$$ 

When it is needed, we also use the notation $\|T\|_{X^*}$. With this norm, the adjoint space $X^*$ becomes a Banach space. We define the scalar product $(u, f) = f[u]$ for every $f \in X^*$ and $u \in X$.

We denote a set of bounded linear operators from $X$ to $Y$ with the finite operator norm $\mathcal{B}(X, Y)$. In case operator $T$ acts on $X$ to itself, we denote $T \in \mathcal{B}(X)$. 
The adjoint space $X^{**}$ of $X^*$ is again a Banach space. We can regard each element $u \in X$ as an element of $X^{**}$. In general, for each $f \in X^*$, there exists some semilinear forms $F[f]$ on $X^*$ that cannot be expressed in the form of $(f, u)$ with $u \in X$. In this sense, $X^{**}$ cannot be identified with $X$ in general. If, however, there is no such $F$, $X$ is called reflexive. In that case, we can identify $X$ with $X^{**}$.

The following facts are well known.

**Lemma 3.1.** For $1 \leq p < \infty$, the adjoint of $L_p(G)$ is identified with $L_q(G)$ with $q$ satisfying $1/p + 1/q = 1$.

**Lemma 3.2.** For $1 < p < \infty$, $L_p(G)$ is reflexive.

**Lemma 3.3.** The adjoint of $C(G)$ is identified with $BV(G)$.

### 4. Definition of cut-norm and known results

For a graphon $W(x, y)$, its cut norm is defined as follows. For a graphon $W$,

$$
\|W\|_{cut} \equiv \sup_{S,T \subset I} \left| \int_{S \times T} W(x, y) \, dx \, dy \right|.
$$

Here, the supremum is taken over all measurable subsets $S$ and $T$ of $I$. The following claim is well known (Lemma 8.10 in [9]).

**Lemma 4.1.** For any non-negative measurable function $W(x, y) : I \times I \to I$, the cut norm $\|W\|_{cut}$ and the quantity

$$
\sup_{f,g: I \to I} \left| \int_I \int_I W(x, y) f(x) g(y) \, dx \, dy \right| \quad (4.1)
$$

are attained and are equal. Here, $f$ and $g$ belong to $L_\infty(I)$, with a range of $I$.

**Remark 4.1.** Here, we do not assume the symmetry of $W$.

The proof of this lemma for a graphon has been given by Lovász [9], but we will simplify it. Next, we define the operator $T_W : L_\infty(I) \to L_1(I)$ by
\[ T_W f \equiv \int_I W(x, y)f(y) \, dy. \]

Following [9], the operator norm of this operator is denoted as \( \|T_W\|_{\infty \to 1} \).

5. Main result

As aforementioned, we improve the inequality concerning the equivalence of the cut norm and operator norm \( \|T_W\| \). Our main result is as follows.

**Theorem 5.1.** For a non-negative measurable function \( W(x, y) \), we assume the following issues with positive constants \( M' \) and \( M'' \):

(i) \( \int_I W(x, y) \, dy \leq M' \) \( \forall x \in I \),

(ii) \( \int_I W(x, y) \, dx \leq M'' \) \( \forall y \in I \).

Then, for arbitrary \( 1 \leq p < \infty \), we can define an integral operator \( T_W u \equiv \int_I W(x, y)u(y) \, dy \) on \( L_p(I) \). This can be regarded as the maximal integral operator \( L_p(I) \rightarrow L_p(I) \) with \( 1 \leq p < \infty \) and \( 1/p + 1/q = 1 \). If we denote its operator norm as \( \|T_W\| \), we have

\[
\|W\|_{\text{cut}} = \|T_W\|. \quad (5.1)
\]

**Remark 5.1.** Lovász [9] regards \( T_W \) as an operator from \( L_\infty(I) \) to \( L_1(I) \). Under the assumption \( \int_I \int_I W(x, y) \, dx \, dy < \infty \), it is maximal as an operator from \( L_\infty(I) \) to \( L_1(I) \). However, this is not the case under the conditions in Theorem 5.1. Besides, the original inequality in [9] reads

\[
\|W\|_{\text{cut}} \leq \|T_W\|_{\infty \to 1} \leq 4\|W\|_{\text{cut}}. \quad (5.2)
\]

We have enhanced the right-hand inequality in (5.2). In addition, our proof does not assume the symmetry of \( W(x, y) \) and can be applied to a digraphon.

**Remark 5.2.** When we think of an adjoint operator \( T^* : Y^* \rightarrow X^* \) of an operator \( T : X = L_\infty(I) \rightarrow Y = L_1(I) \), note that \( Y^* = (L_1(I))^* = L_\infty(I) \) and \( X^* = (L_\infty(I))^* = (L_1(I))^{**} \). In general, this space does not match with \( L_1(I) \), but instead includes it as a subset. Since \( C(I) \subset L_\infty(I) \), we have \( (L_\infty(I))^* \subset BV(I) \). Thus, the range of the operator \( T_W \) in this case is not
$L_1(I)$ itself but includes $L_1(I)$ as its subset. Actually, $(L_\infty(I))^*$ is identified with the space of a Radon measure [2].

However, in the context of graphons and digraphons, the adjoint operator of $T_W$ is from $L_\infty(I)$ to $L_1(I)$ as we is shown later.

**Remark 5.3.** Janson [5] regarded the graphon as an operator from $L_p(\Omega)$ to $L_q(\Omega)$ for a general probability space $\Omega$ (see, Appendix E.) The author also derived the inequality

$$\|W\|_{cut} = \|T_W\|_{\infty,1} \leq \|T_W\|_{p,q} \leq \sqrt{2}\|W\|_{cut}^{\min\{1/p,1/q\}},$$

where $\|T_W\|_{p,q}$ is an operator norm of $T_W$ as an operator from $L_p(I)$ to $L_q(I)$.

**Remark 5.4.** The assumptions (i) and (ii) in Theorem 5.1 are stronger than $\int_I \int_I W(x,y) \, dx \, dy < \infty$.

**Corollary 5.1.** Under the assumptions of Theorem 5.1, the operator $T_W$ is defined on $L_\infty(I)$ to itself, and the equality (5.1) holds again.

### 6. Proofs

#### 6.1. Proof of Lemma 4.1

We should prove two things: First, we show the equivalence of $\|W\|_{cut}$ and (4.1). Second, we verify that the supremum of these quantities is certainly attained by some $f, g \in L_\infty(I)$.

To both of these proofs, we apply the standard theory on Banach spaces. For simplicity, we set $X = L_\infty(I)$ and $Y = L_1(I)$. Then, it is clearly seen that $T_W \in \mathcal{B}(X,Y)$ under the condition $\|W\|_{cut} < \infty$.

Thanks to the well known facts found in [7], we have the expression for the operator norm $\|T_W\|$ for $T_W \in \mathcal{B}(X,Y)$ (see pp.26 and 150 in [7]):

$$\|T_W\| = \sup_{0 \neq f \in X, 0 \neq g \in Y^*} \frac{|(T_Wf, g)|}{\|f\|_X \|g\|_{Y^*}} = \sup_{\|f\|_X = 1, \|g\|_{Y^*} = 1} |(T_Wf, g)|. \quad (6.1)$$
Since \((L_1(I))^* = L_\infty(I)\), it is obvious that \(\|T_W\|_{\infty \to 1}\) is equal to (6.1). Thanks to the proof of Theorem 5.1 below, we then have the equivalence of \(\|W\|_{\text{cut}}, \|W\|_{\infty \to 1}\) and (4.1).

Next, we take a sequence \((f_n, g_n) \in L_\infty(I) \times L_\infty(I)\), such that \(\|f_n\|_X = \|g_n\|_{Y^*} = 1\) and \((T_W f_n, g_n) \to \|T_W\|\).

Notice that in (6.1), \(g \in Y^* = L_\infty(I)\). Applying the weakly-* compactness known as the Banach–Alaoglu–Bourbaki’s theorem [2], we have \((u, g_n) \to (u, g_0)\) for arbitrary \(u \in Y\). Especially, we have \((T_W f_k, g_n) \to (T_W f_k, g_0)\) for each \(k = 1, 2, \ldots\). Then, it is clear that \((T_W f_k, g_0) \to \|T_W\|\) as \(k \to \infty\). Actually, for arbitrarily small \(\varepsilon > 0\), if we take \(K_0\) sufficiently large, then for \(k \geq K_0\),

\[
\left| (T_W f_k, g_0) - \|T_W\| \right| \leq \left| (T_W f_k, g_0) - (T_W f_k, g_k) \right| + \left| (T_W f_k, g_k) - \|T_W\| \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.
\]

On the other hand, it is important to note that the adjoint operator of \(T_W\), denoted as \(T_W^*\) hereafter, is defined as

\[
T_W^* g(x) = \int_I W(y, x) g(y) \, dy.
\]

In the case of a graphon, \(T_W^* = T_W\) due to its symmetry. Regardless, under the assumption of Lemma 4.1, it is also an operator from \(L_\infty(I)\) to \(L_1(I)\). Thus, we have \((f_k, T_W^* g_0) \to \|T_W\|\). Since \(\{f_k\}\) is a bounded sequence in \(L_\infty(I)\), again due to the weakly-* compactness, we have \((f_k, T_W^* g_0) \to (f_0, T_W^* g_0)\), with a certain \(f_0 \in X = L_\infty(I)\). This means \((f_k, T_W^* g_0) \to \|T_W\|\), and thereby \((T_W f_n, g_n) \to (T_W f_0, g_0) = \|T_W\|\).

### 6.2. Proof of Theorem 5.1

Now, we show the proof of Theorem 5.1. Note that in this case, \(T_W\) is regarded as an operator from \(X = L_p(I)\) (\(p \in (1, \infty)\)) to itself. Now, the inequality

\[
\|W\|_{\text{cut}} \leq \sup_{f, g: I \to I} \left| \int_I \int_I W(x, y) f(x) g(y) \, dx \, dy \right|
\]

obviously holds, since the left-hand side is a special case of the right-hand side, with \(f(x) = 1_I(x)\) and \(g(y) = 1_I(y)\).

Next, we will discuss the opposite inequality. From the standard argument, using \(q\) that satisfies \(1/p + 1/q = 1\), we have

\[
\int_I W(x, y) f(y) \, dy
\]
\[
\begin{align*}
&= \int_I |W(x,y)|^{1-\frac{1}{p}} |W(x,y)|^{\frac{1}{p}} |f(y)| dy \\
&\leq \left( \int_I |W(x,y)| \, dy \right)^{\frac{1}{q}} \left( \int_I |W(x,y)||f(y)| \, dy \right)^{\frac{1}{p}} \\
&\leq (M')^{\frac{p-1}{p}} \left( \int_I |W(x,y)||f(y)| \, dy \right)^{\frac{1}{p}}.
\end{align*}
\]

Thus, we have
\[
\begin{align*}
\int_I \left| \int_I W(x,y)f(y) \, dy \right|^p \, dy \\
&\leq (M')^{p-1} \int_I dx \left( \int_I |W(x,y)||f(y)|^p \, dy \right) \\
&\leq (M')^{p-1} \int_I |f(y)|^p \left( \int_I |W(x,y)| \, dx \right) \, dy \\
&\leq (M')^{p-1} M'' \|f\|_{L^p(I)}^p.
\end{align*}
\]

Here, we utilize the Fubini theorem, which is allowed since \(W(x,y) \geq 0\) is absolutely integrable. Thus, we have
\[
\|T_w\| \leq (M')^{\frac{p-1}{p}} (M'')^{\frac{1}{p}}.
\]

From the definition of \(M'\) and \(M''\), we have \(\max\{M', M''\} \leq \|W\|_{cut}\). Thus, we have arrived at \(\|T_w\| \leq \|W\|_{cut}\), which yields the desired result. The proof of Corollary 5.1 is obvious so it is left out of this note.

7. Conclusions

In this note, we have observed two things: First, by making use of the arguments of functional analysis, we have simplified the proof of Lemma 4.1 by Lovász; second, we have improved the known relationship between the cut norm and operator norm of non-negative measurable functions. The arguments put forth by Lovász were limited to only symmetric kernels, whereas ours are applicable to non-symmetric measurable functions. For instance, our results are useful when discussing the sampling method from a digraphon to make a suitable network for a certain kind of machine learning [4].

Since our purpose is to shorten the proof of existing results, this article is very brief. Given our results, the equivalence of the cut norm and operator norm
norm remains the same. However, our results, especially Theorem 5.1, are useful for improving the estimate of norms in future works concerning graphons and digraphons.

References


